Observer-Based Stabilization of Some Nonlinear Non-Minimum Phase Systems Using Neural Network

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Abstract: This paper presents a neuro-adaptive output-feedback stabilization method for nonlinear non-minimum phase systems with partially known Lipschitz continuous functions in their arguments. The proposed controller is comprised of a linear, a neuro-adaptive, and an adaptive robustifying control term. The adaptation laws for the neural network weights are obtained using the Lyapunov's direct method. These adaptation laws employ a suitable output of a linear state observer that is realizable. The ultimate boundedness of the error signals will be shown through analytical work using Lyapunov's method. The effectiveness of the proposed scheme will be shown in simulations for the benchmark Single Flexible Link Manipulator and Translation Oscillator Rotational Actuator (TORA) problems.

Keywords: neuro-adaptive control; nonlinear systems; non-minimum phase; output feedback.

1 Introduction

Output-feedback control of nonlinear systems is a challenging problem in the control theory. This problem has been an active research area for many years. Several fundamental methods have been proposed in this area including geometric techniques (Isidori, 1995), adaptive observers and output-feedback controllers for system in output-feedback form (Marino and Tomi, 1995), high gain observers (Khalil, 2002), and back-stepping method for systems with parametric uncertainties (Krstic et al., 1995). The aim of all these research efforts is to develop a systematic design method for controlling systems in presence of structured uncertainties in the form of parameters variations and unstructured uncertainties such as unmodeled dynamics. Due to the universal approximation properties of NN, these methods are applied to a greater general class of systems with structured and unstructured uncertainties. Remarkably, these results include methods for uncertain systems based on the high-gain observer (Ge and Zhang, 2003; Seshagiri and Khalil, 2000) and adaptive error observer (Hovakimyan et. al, 2002). A common assumption in the aforementioned methods is that the zero dynamics are globally asymptotically stable or input-to-state stable (ISS). In other words, the system is assumed minimum phase.

Recently, some papers have dealt with output-feedback stabilization for uncertain non-minimum phase systems. Isidori (2000) has proposed a solution for robust semi-global output-feedback stabilization of non-minimum phase systems based on auxiliary constructions using a high-gain observer. Karagiannis et al. (2005) and Wang (2008) have achieved global output-feedback stabilization using the classical back stepping and the small-gain techniques. A design method for semi-global stabilization of a class of non-minimum phase nonlinear systems, which can be transformed to the global normal form as well as to the form of linear observer error dynamic, has been presented by Ding (2005). Yang (2006) has applied the sliding-mode observer and the output-feedback sliding-mode control. An important note is that these methods can only be applied to systems, where nonlinearities or high frequency gains depend only on the output of the system. Moreover, zero assignment by redefining the output of system (Kazantizis and Niemiec, 2004), and stable inversion using the iterative learning control method (Norrlof and Gunnarsson, 2001) are proposed in literatures to control non-minimum phase systems. However, these methods are limited to systems with known dynamics.

To relax some required information on the system model, several methods have been presented base on the approximation property of NNs. The local and non-local stabilization methods for uncertain non-minimum phase systems with unstructured uncertainties have been presented recently by some researchers (Chen and Chen 2003; Lee, 2004; Hoseini and Farrokhi, 2007). However, these methods are based on the state feedback.

This paper presents an adaptive output-feedback stabilization method for observable and stabilizable nonlinear non-minimum phase systems, which can be either affine or non-affine. Moreover, only an approximate linear model of the nonlinear system is required in the design procedure. This linear system should present the non-minimum phase zeros of the nonlinear system with desired accuracy. In fact, there is a conic sector bound on the modelling error of the non-minimum phase zeros that is referred to as the unmatched uncertainty. Hence, the proposed approach can be applied to uncertain systems, which have partially known Lipschitz continuous functions in their arguments. In addition, the uncertain nonlinearities of the system do not need to be restricted only to the output of the system.

In the design procedure, first, an optimal linear controller is proposed that stabilizes the linear part of dynamics. Then, this linear controller is augmented with a neuro-adaptive term, which approximates the matched uncertainty. The NN operates over a tapped-delay units comprised of the system input-output signals. In addition, an adaptive robustifying control term is added to the control law to compensate the NN approximation error. Moreover, a suitable linear observer is design such that the combined control law depends only on the output of the system.

This paper is organized as follows: Section 2 describes the class of nonlinear systems and defines the problem of stabilization. Section 3 presents the design procedure for the controller and the observer design and approximation properties of the NN. Section 4 shows the stability analysis of the closed-loop system. Section 5 illustrates simulation examples for a flexible-link manipulator as well as for the TORA system. Finally, Section 6 concludes the paper.

2 Problem statement

Consider the nonlinear SISO system in the following normal form with the coordinates $\left[\mathbf{z}^T,\boldsymbol{\eta}^T\right]=\left[z_1,\ldots,z_r,\eta_{r+1},\ldots,\eta_n\right]$ (Marino and Tomi 1995):

$$\begin{cases}\dot{z}_i=z_{i+1} & 1\le i\le r-1\\ \dot{z}_r=f\left(\mathbf{z},\boldsymbol{\eta},u\right)\\ \dot{\boldsymbol{\eta}}=\mathbf{v}\left(\mathbf{z},\boldsymbol{\eta}\right)\\ y=z_1,\end{cases}\tag{1}$$

where r is the relative degree, $\boldsymbol{\eta}\in\Omega_\eta\subset\mathbb{R}^{n-r}$ is the state vector associated with the internal dynamics, $\mathbf{z}=\left[z_1,\cdots,z_r\right]^T\in\Omega_z\subset\mathbb{R}^r$, Ω_η and Ω_z are the compact sets of the operating regions, and $u\in\mathbb{R}$ and $y\in\mathbb{R}$ are the input and the output of the system, respectively. The mappings $f:\mathbb{R}^{n+1}\to\mathbb{R}$ and $\mathbf{v}:\mathbb{R}^n\to\mathbb{R}^{n-r}$ are partially known Lipschitz continuous functions of their arguments with initial conditions $f\left(\mathbf{0},\mathbf{0},0\right)=0$ and $\mathbf{v}\left(\mathbf{0},\mathbf{0}\right)=\mathbf{0}$. Note that the system in (1) can be non-minimum phase; hence, there is no need for any assumption on the stability of the zero dynamics. Moreover, it is assumed that the system output y is measurable. The goal is to design a controller to stabilize all state variables including internal dynamics of the system. The various features of the proposed control design scheme are presented in the next section.

3 Controller design

3.1 Model expansion

Using the Taylor expansion method, the system in (1) can be expressed around its equilibrium point at the origin as

$$\begin{cases}\dot{z}_i=z_{i+1} & 1\le i\le r-1\\ \dot{z}_r=\mathbf{m}^T\mathbf{z}+\mathbf{n}^T\boldsymbol{\eta}+\psi\left(\mathbf{z},\boldsymbol{\eta},u\right)\\ \dot{\boldsymbol{\eta}}=\mathbf{F}\,\boldsymbol{\eta}+\mathbf{G}\,\mathbf{z}+\boldsymbol{\Delta}_{\boldsymbol{\eta}}\left(\mathbf{z},\boldsymbol{\eta}\right)\\ y=z_1,\end{cases}\tag{2}$$

where $\mathbf{m}$ and $\mathbf{n}$ are coefficient vectors, and $\mathbf{F}$ and $\mathbf{G}$ are matrices, all with appropriate dimensions. In addition, $\boldsymbol{\Delta}_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})$ denotes the vector of the zero-dynamic modelling error or the unmatched uncertainties.

Assumption 1: *The unmatched uncertainties are bounded in the operating region as*

$$\left\|\boldsymbol{\Delta}_{\boldsymbol{\eta}}(\mathbf{z},\boldsymbol{\eta})\right\|\le c_1\left\|\mathbf{z}\right\|+c_2\left\|\boldsymbol{\eta}\right\|\quad\forall\left(\mathbf{z},\boldsymbol{\eta}\right)\in\Omega_z\times\Omega_\eta,\tag{3}$$

where c_1 and c_2 are known positive constants.

Let $\hat{\psi}\left(y,u\right)$ be the best available approximation of $\psi\left(\mathbf{z},\boldsymbol{\eta},u\right)$. Consider the pseudo control

$$\upsilon=\hat{\psi}\left(y,u\right).\tag{4}$$

Then, the matched modelling error is

$$\Delta\left(\mathbf{z},\boldsymbol{\eta},u\right)=\psi\left(\mathbf{z},\boldsymbol{\eta},u\right)-\hat{\psi}\left(y,u\right)\tag{5}$$

Now, let define $\boldsymbol{\xi}\triangleq\left[\mathbf{z}^T,\boldsymbol{\eta}^T\right]^T$ and select the pseudo control as

$$\upsilon=u_L-u_{ad}-u_R.\tag{6}$$

Using (4), (5) and (6), the system in (2) can be described as

$$\begin{cases} \dot{\boldsymbol{\xi}} = \mathbf{A}\boldsymbol{\xi} + \mathbf{b}u_L + \mathbf{b}\left(\Delta - u_{ad} - u_R\right) + \mathbf{H}\boldsymbol{\Delta}_{\boldsymbol{\eta}} \\ y = \mathbf{c}\boldsymbol{\xi}, \end{cases} \tag{7}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{M} & \mathbf{N} \\ \mathbf{G} & \mathbf{F} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} \mathbf{0}_{1\times(r-1)} & 1 & \mathbf{0} \end{bmatrix}^T, \mathbf{H} = \begin{bmatrix} \mathbf{0}_{(n-r)\times r} & \mathbf{I}_{(n-r)\times(n-r)} \end{bmatrix}^T$$

$$\mathbf{c} = \begin{bmatrix} 1 & \mathbf{0}_{1\times(n-1)} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} \mathbf{0}_{(r-1)\times 1} & \mathbf{I}_{(r-1)\times(r-1)} \\ \hline \mathbf{m}^T & \end{bmatrix}, \mathbf{N} = \begin{bmatrix} \mathbf{0}_{(r-1)\times(n-r)} \\ \mathbf{n}^T \end{bmatrix}$$

Note that, since the system is non-minimum phase, some eigenvalues of $\mathbf{A}$ are positive. Moreover, it is important to point out that the model approximation function $\hat{\psi}(\cdot,\cdot)$ should be defined such that it is invertible with respect to u, allowing the actual control input to be computed by

$$u = \hat{\psi}^{-1}\left(y, \upsilon\right).$$

3.2 Linear controller design

Consider the linear part of the system in (7). It is well known that controllability of $(\mathbf{A}, \mathbf{b})$ ensures existence of a linear optimal control law as

$$u_L = -\mathbf{k}_c\hat{\boldsymbol{\xi}} \equiv -\frac{1}{\gamma}\rho, \tag{8}$$

which minimizes $\int_0^\infty \left(\boldsymbol{\xi}^T\mathbf{Q}_1\boldsymbol{\xi} + \gamma u^2\right)dt$, where $\mathbf{Q}_1$ is an arbitrary symmetric positive-definite matrix, γ is a given positive number and $\hat{\boldsymbol{\xi}}$ denotes the estimation of $\boldsymbol{\xi}$. The vector gain $\mathbf{k}_c$ is derived as

$$\mathbf{k}_c^T = \frac{1}{\gamma}\mathbf{P}_1\mathbf{b}, \tag{9}$$

where $\mathbf{P}_1 = \mathbf{P}_1^T > 0$ is the solution of the following algebraic Riccati equation:

$$\mathbf{P}_1\mathbf{A} + \mathbf{A}^T\mathbf{P}_1 + \mathbf{Q}_1 - \frac{2}{\gamma}\mathbf{P}_1\mathbf{b}\mathbf{b}^T\mathbf{P}_1 = 0 \tag{10}$$

Substituting (9) into (10) gives

$$\left(\mathbf{A} - \mathbf{b}\mathbf{k}_c\right)^T\mathbf{P}_1 + \mathbf{P}_1\left(\mathbf{A} - \mathbf{b}\mathbf{k}_c\right) + \mathbf{Q}_1 = 0. \tag{11}$$

Hence, $\mathbf{A} - \mathbf{b}\mathbf{k}_c$ is a stable matrix and u_L can stabilize the system when it is linear.

3.3 Neuro-adaptive control design

The adaptive part of the control law u_{ad} in (6) is designed to approximate $\Delta(\mathbf{z}, \boldsymbol{\eta}, u)$. Hence, there exists a fixed-point problem as

$$u_{ad}(t) = \Delta\left(\mathbf{z}, \boldsymbol{\eta}, \hat{\psi}^{-1}\left(y, u_L - u_R - u_{ad}\right)\right). \tag{12}$$

The following assumption provides conditions, which guarantee existence and uniqueness of a solution for u_{ad}.

Assumption 2: *The map $u_{ad} \rightarrow \Delta$ is a contraction over the entire input domain. This means, the following inequality should be satisfied*

$$\left|\frac{\partial\Delta}{\partial u_{ad}}\right| < 1. \tag{13}$$

Substituting (4), (5) and (6) into (13) yields

$$\left|\frac{\partial \Delta}{\partial u_{ad}}\right| = \left|\frac{\partial\left(\psi\left(\mathbf{z},\boldsymbol{\eta},u\right)-\hat{\psi}\left(y,u\right)\right)}{\partial u}\frac{\partial u}{\partial v}\frac{\partial v}{\partial u_{ad}}\right|$$
$$= \left|-\frac{\partial\left(\psi-\hat{\psi}\right)}{\partial u}\frac{\partial u}{\partial \hat{\psi}}\right| < 1. \tag{14}$$

It can be easily verified that if the following three conditions are satisfied then condition (14) is ensured:

$$\operatorname{sgn}\left(\frac{\partial \psi}{\partial u}\right) = \operatorname{sgn}\left(\frac{\partial \hat{\psi}}{\partial u}\right), \left|\frac{\partial \hat{\psi}}{\partial u}\right| > 0.5\left|\frac{\partial \psi}{\partial u}\right|, \frac{\partial \psi}{\partial u} \neq 0 \tag{15}$$

The third condition implies that the smooth function $\psi_u = \partial\psi/\partial u$ is strictly either positive or negative on the compact set $\Omega_z \times \Omega_\eta \times R$.

Remark 1: *When the exact value of coefficient matrices in the model expansion (2) are not available (e.g. in the presence of parameters uncertainties), the estimated values* $\hat{\mathbf{m}}, \hat{\mathbf{n}}, \hat{\mathbf{F}}$ *and* $\hat{\mathbf{G}}$ *may be used to design the linear controller. In this case, the modelling errors arise from the difference between the real and the estimated value of coefficients can be embodied in* Δ *and* $\boldsymbol{\Delta}_{\boldsymbol{\eta}}$ *while Assumptions 1 and 2 are satisfied.*

Under the observability condition of system in (1), it has been shown by Lavertsky et al. (2003) that the continuous-time dynamic $\Delta(\mathbf{z}, \boldsymbol{\eta}, u)$ can be approximated using the delayed version of inputs and outputs as

$$\Delta(\mathbf{z}, \boldsymbol{\eta}, u) = \Gamma\left(\boldsymbol{\zeta}\right) + \varepsilon_0\,, \tag{16}$$

where $\boldsymbol{\zeta} = \begin{bmatrix}1 & \bar{\mathbf{y}} & \bar{\mathbf{u}}\end{bmatrix}^T \in \mathbb{R}^N$, $\Gamma : R^N \to R$ and

$$\begin{aligned}\bar{\mathbf{u}} &= \begin{bmatrix}u(t-T_d) & \cdots & u(t-T_d(n_1-r-1))\end{bmatrix},\ n_1 \geq n\\ \bar{\mathbf{y}} &= \begin{bmatrix}y(t) & \cdots & y(t-T_d(n_1-1))\end{bmatrix}\end{aligned}$$

The approximation error ε_0 is directly proportional to the sampling time interval T_d. Hence, it can be ignored by selecting T_d sufficiently small.

On the other hand, any sufficiently smooth function can be approximated on a compact set with an arbitrarily bounded error by a suitable large MLP (Hornik, et al. 1989). Therefore, a set of ideal weights, denoted as $\mathbf{w}^*$ and $\mathbf{V}^*$, exist on the compact set Ω_Δ such that

$$\Delta\left(\mathbf{z},\boldsymbol{\eta},u\right) = \mathbf{w}^{*T}\boldsymbol{\sigma}(\mathbf{V}^{*T}\boldsymbol{\zeta}) + \varepsilon_1 \qquad \forall \boldsymbol{\zeta} \in \Omega_\Delta\,, \tag{17}$$

where $\mathbf{w}^* \in \mathbb{R}^m$ is a vector containing the synaptic weights connecting the hidden layer to the output layer, $\mathbf{V}^* \in R^{N\times m}$ is a matrix containing the synaptic weights connecting the input layer to the hidden layer, $\boldsymbol{\sigma} = \begin{bmatrix}\sigma_1 \cdots \sigma_m\end{bmatrix}^T$ is a vector function containing the nonlinear functions of the neurons in the hidden layer, and $|\varepsilon_1| \leq \varepsilon_{1M}$, in which ε_{1M} depends on the network architecture. The ideal constant weights $\mathbf{w}^*$ and $\mathbf{V}^*$ are defined as

$$\left(\mathbf{w}^*, \mathbf{V}^*\right) \triangleq \arg \min_{(\mathbf{w},\mathbf{V})\in\Omega_{\mathrm{w}}} \left\{\sup_{\boldsymbol{\zeta}\in\Omega_\zeta}\left|\mathbf{w}^T\boldsymbol{\sigma}\left(\mathbf{V}^T\boldsymbol{\zeta}\right) - \Gamma(\boldsymbol{\zeta})\right|\right\} \tag{18}$$

where $\Omega_{\mathrm{w}} = \left\{(\mathbf{w},\mathbf{V})\middle| \|\mathbf{w}\|_{\mathrm{F}} \leq M_{\mathrm{w}}, \|\mathbf{V}\|_{\mathrm{F}} \leq M_{\mathrm{V}}\right\}$, M_{w} and M_{V} are positive numbers, and $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. Since Δ can be approximated with a NN, a Multi-Layer Perceptron (MLP) is employed to construct the adaptive control term as

$$u_{ad} = \mathbf{w}^T\boldsymbol{\sigma}\left(\mathbf{V}^T\boldsymbol{\zeta}\right). \tag{19}$$

In practice, the weights of this neural network may be different from the ideal ones, defined in (18). Therefore, an approximation error exists.

Lemma 1: *Let basis function* $\sigma(\cdot)$ *be logistic or hyperbolic tangent. Then, the approximation error, which arises from the difference between (17) and (19), satisfies the following equality:*

$$\Delta(\mathbf{z},\boldsymbol{\eta},u)-u_{ad}=\tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma}-\dot{\boldsymbol{\sigma}}\mathbf{V}^T\zeta\right)+\mathrm{tr}\left(\tilde{\mathbf{V}}^T\zeta\mathbf{w}^T\dot{\boldsymbol{\sigma}}\right)+\delta(t) \tag{20}$$

where

$$\left.\begin{aligned}&|\delta(t)|\le\left(\varepsilon_{1M}+2\sqrt{m}M_{\mathrm{w}}\right)+M_{\mathrm{w}}\left\|\tilde{\mathbf{V}}\right\|_{\mathrm{F}}\|\zeta\|+M_{\mathrm{V}}\|\tilde{\mathbf{w}}\|\|\zeta\|,\\&\tilde{\mathbf{w}}=\mathbf{w}^*-\mathbf{w},\\&\tilde{\mathbf{V}}=\mathbf{V}^*-\mathbf{V},\end{aligned}\right\} \tag{21}$$

and $\dot{\boldsymbol{\sigma}}=\mathrm{diag}\left[\partial\sigma_1(v_1)/v_1 \quad \ldots \quad \partial\sigma_m(v_m)/v_m\right]$ *is the derivative of* $\boldsymbol{\sigma}$ *with respect to the input signals* v_i $(i=1,\ldots,m)$*, in which* $[v_1,\ldots,v_m]^T=\mathbf{V}^T\boldsymbol{\zeta}$*, and m denotes the number of neurons in the hidden layer.*

Proof: Using the Taylor series expansion of $\boldsymbol{\sigma}(\mathbf{V}^{*T}\zeta)$, it yields

$$\begin{aligned}\boldsymbol{\sigma}(\mathbf{V}^{*T}\zeta)&=\boldsymbol{\sigma}(\mathbf{V}^T\zeta+\tilde{\mathbf{V}}^T\zeta)\\&=\boldsymbol{\sigma}(\mathbf{V}^T\zeta)+\dot{\boldsymbol{\sigma}}(\mathbf{V}^T\zeta)\tilde{\mathbf{V}}^T\zeta+\mathbf{O}(\cdot),\end{aligned} \tag{22}$$

where $\mathbf{O}(\cdot)\in R^m$ denotes the vector associated with high order terms.

Notice that the activation function in neurons of MLP is of sigmoid type functions (e.g. logistic function $\dfrac{1}{1+e^{-\alpha v_i}}$ or $\tanh(\alpha v_i)$). Hence, $|\sigma_i|\le 1$ and $|\partial\sigma_i(v_i)/v_i|\le 1$. Consequently, it is easy to show that $\|\boldsymbol{\sigma}\|\le\sqrt{m}$ and $\|\dot{\boldsymbol{\sigma}}\|\le\alpha$. Using these inequalities and (22), it can be concluded that $\|\mathbf{O}(\cdot)\|$ is also bounded and can be represented as

$$\begin{aligned}\|\mathbf{O}(\cdot)\|&=\left\|\boldsymbol{\sigma}(\mathbf{V}^{*T}\zeta)-\boldsymbol{\sigma}(\mathbf{V}^T\zeta)-\dot{\boldsymbol{\sigma}}(\mathbf{V}^T\zeta)\tilde{\mathbf{V}}^T\zeta\right\|\\&\le 2\sqrt{m}+\alpha\left\|\tilde{\mathbf{V}}\right\|_{\mathrm{F}}\|\zeta\|.\end{aligned} \tag{23}$$

Therefore, the approximation error can be calculated as

$$\begin{aligned}\Delta-u_{ad}&=\mathbf{w}^{*T}\boldsymbol{\sigma}(\mathbf{V}^{*T}\zeta)+\varepsilon_1-\mathbf{w}^T\boldsymbol{\sigma}(\mathbf{V}^T\zeta)\\&=\left(\mathbf{w}^T+\tilde{\mathbf{w}}^T\right)\left(\boldsymbol{\sigma}(\mathbf{V}^T\zeta)+\dot{\boldsymbol{\sigma}}(\mathbf{V}^T\zeta)\left(\mathbf{V}^{*T}-\mathbf{V}^T\right)\zeta+\mathbf{O}(\cdot)\right)\\&\quad+\varepsilon_1-\mathbf{w}^T\boldsymbol{\sigma}(\mathbf{V}^T\zeta)\\&=\tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma}-\dot{\boldsymbol{\sigma}}\mathbf{V}^T\zeta\right)+\mathbf{w}^T\dot{\boldsymbol{\sigma}}\tilde{\mathbf{V}}^T\zeta+\tilde{\mathbf{w}}^T\dot{\boldsymbol{\sigma}}\mathbf{V}^{*T}\zeta+\mathbf{w}^{*T}\mathbf{O}(\cdot)+\varepsilon_1\\&=\tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma}-\dot{\boldsymbol{\sigma}}\mathbf{V}^T\zeta\right)+\mathrm{tr}\left(\tilde{\mathbf{V}}^T\zeta\mathbf{w}^T\dot{\boldsymbol{\sigma}}\right)+\delta,\end{aligned}$$

where

$$\delta\triangleq\tilde{\mathbf{w}}^T\dot{\boldsymbol{\sigma}}\mathbf{V}^{*T}\zeta+\mathbf{w}^{*T}\mathbf{O}(\cdot)+\varepsilon_1\,.$$

Now, using (18) and (23), and the fact that $\|\dot{\boldsymbol{\sigma}}\|\le\alpha$, it gives

$$\begin{aligned}|\delta|&\le\|\tilde{\mathbf{w}}\|\|\dot{\boldsymbol{\sigma}}\|\left\|\mathbf{V}^*\right\|_F\|\zeta\|+\left\|\mathbf{w}^*\right\|\|\mathbf{O}\|+\varepsilon_{1M}\\&\le\alpha M_{\mathrm{V}}\|\tilde{\mathbf{w}}\|\|\zeta\|+M_{\mathrm{w}}\left(2\sqrt{m}+\alpha\left\|\tilde{\mathbf{V}}\right\|_F\|\zeta\|\right)+\varepsilon_{1M}\\&=\left(\varepsilon_{1M}+2\sqrt{m}M_{\mathrm{w}}\right)+\alpha M_{\mathrm{w}}\left\|\tilde{\mathbf{V}}\right\|_{\mathrm{F}}\|\zeta\|+\alpha M_{\mathrm{V}}\|\tilde{\mathbf{w}}\|\|\zeta\|\end{aligned}$$

□

The adaptation rules for the weights of the neuro-adaptive controller, given in (19), is proposed as

$$\left.\begin{aligned}\dot{\mathbf{w}}&=\gamma_w\left(\rho\left(\boldsymbol{\sigma}-\dot{\boldsymbol{\sigma}}\mathbf{V}^T\zeta\right)-k_w\mathbf{w}\right)\\\dot{\mathbf{V}}&=\gamma_V\left(\rho\,\zeta\mathbf{w}^T\dot{\boldsymbol{\sigma}}-k_V\mathbf{V}\right)\end{aligned}\right\}, \tag{24}$$

where ρ is introduced in (9), γ_{w} and γ_{V} are the learning coefficients, and k_w and k_V are the σ-modification gains, which are used to avoid the persistence excitation condition of the inputs to the NN and to guarantee the boundedness of $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{V}}$ (Ioannou and Kokotovic, 1983; Lewis, et al., 1996).

Note that the following conservative upper bound of the approximation error can be obtained by using (17), (19) and the fact that $\|\boldsymbol{\sigma}\| \le \sqrt{m}$:

$$
\begin{aligned}
\left|\Delta(\mathbf{z},\boldsymbol{\eta},u)-u_{ad}\right| &= \left|\mathbf{w}^{*T}\boldsymbol{\sigma}\left(\mathbf{V}^{*T}\zeta\right)+\varepsilon_1-\mathbf{w}^T\boldsymbol{\sigma}\left(\mathbf{V}^T\zeta\right)\right| \\
&\le 2\sqrt{m}\ M_{\mathrm{w}}+\varepsilon_{1M}.
\end{aligned}
\tag{25}
$$

3.4 *The adaptive robustifying control design*

The neuro-adaptive control term u_{ad}, with adaptation rules in (24), cannot provide exact solution to the matched uncertainty and there still exists an approximation error $\delta(t)$. In order to compensate for this error, an adaptive robustifying term u_R is proposed here. Using (18) and (21), the upper bound of the approximation error can be calculated as

$$
\begin{aligned}
|\delta| &\le \left(\varepsilon_{1M}+2\sqrt{m}M_{\mathrm{w}}\right)+\alpha M_{\mathrm{V}}\ \left\|\mathbf{w}^*-\mathbf{w}\right\|\|\zeta\| \\
&\quad +\alpha M_{\mathrm{w}}\ \left\|\mathbf{V}^*-\mathbf{V}\right\|_F\ \|\zeta\| \\
&\le \left(\varepsilon_{1M}+2\sqrt{m}M_{\mathrm{w}}\right)+\alpha M_{\mathrm{V}}\ \left\|\mathbf{w}^*\right\|\|\zeta\|+\alpha M_{\mathrm{V}}\ \|\mathbf{w}\|\|\zeta\| \\
&\quad +\alpha M_{\mathrm{w}}\ \left\|\mathbf{V}^*\right\|_{\mathrm{F}}\|\zeta\|+\alpha M_{\mathrm{w}}\ \|\mathbf{V}\|_{\mathrm{F}}\ \|\zeta\| \\
&\le \left(\varepsilon_{1M}+2\sqrt{m}M_{\mathrm{w}}\right)+\alpha M_{\mathrm{V}}\ M_{\mathrm{w}}\|\zeta\|+\alpha M_{\mathrm{V}}\ \|\mathbf{w}\|\|\zeta\| \\
&\quad +\alpha M_{\mathrm{w}}\ M_{\mathrm{V}}\ \|\zeta\|+\alpha M_{\mathrm{w}}\ \|\mathbf{V}\|_{\mathrm{F}}\ \|\zeta\| \\
&\le \varphi^*\left(1+\|\zeta\|\left(1+\|\mathbf{V}\|_{\mathrm{F}}+\|\mathbf{w}\|\right)\right)=\varphi^*\chi,
\end{aligned}
\tag{26}
$$

where $\varphi^*=\max\left\{\varepsilon_{1M}+2\sqrt{m}M_{\mathrm{w}}\ ,\alpha M_{\mathrm{w}},\alpha M_{\mathrm{V}}\ ,2\alpha M_{\mathrm{V}}\ M_{\mathrm{w}}\right\}$ and $\chi \triangleq 1+\|\zeta\|\left(1+\|\mathbf{V}\|_{\mathrm{F}}+\|\mathbf{w}\|\right)$. Hence, $\delta(t)$ can be limited to a multiplication of the known function χ and an unknown gain φ^*. The following adaptive robustifying term is introduced to compensate $\delta(t)$:

$$u_R=\chi\ \varphi\ \mathrm{sign}\left(\rho\right), \tag{27}$$

with the following adaptation rule:

$$\dot{\varphi}=\gamma_\varphi\chi\left|\rho\right|, \tag{28}$$

where γ_φ is the learning constant and φ denotes estimation of the unknown gain φ^*.

Note that, due to the universal approximation property of NNs, the approximation error $\delta(t)$ is bounded. Hence, it is always possible to find a positive constant U_M such that

$$\left|u_R\right| \le U_M\ . \tag{29}$$

3.5 *Observer design*

For realization of weight adaptation rules, given in (24) and (28), (i.e. dependence of the controller only to the available data), a linear state estimator is proposed as

$$\dot{\hat{\xi}}=\left(\mathbf{A}-\mathbf{k}_o\mathbf{c}\right)\hat{\xi}+\mathbf{b}\,u_L+\mathbf{k}_o y\ , \tag{30}$$

where $\mathbf{k}_o$ is selected such that $\mathbf{A}-\mathbf{k}_o\mathbf{c}$ is stable. The stability of $\mathbf{A}-\mathbf{k}_o\mathbf{c}$ assures existence of the solution $\mathbf{P}_2=\mathbf{P}_2^T>0$ for the following Riccati equation for some $\mathbf{Q}_2=\mathbf{Q}_2^T>0$:

$$
\begin{aligned}
\mathbf{P}_2\left(\mathbf{A}-\mathbf{k}_o\mathbf{c}\right)+\left(\mathbf{A}-\mathbf{k}_o\mathbf{c}\right)^T\mathbf{P}_2 &= -\mathbf{Q}_2-2\mathbf{k}_c^T\mathbf{k}_c \\
&\quad -\mathbf{c}^T\mathbf{k}_o^T\ \mathbf{P}_1\mathbf{Q}_1^{-1}\mathbf{P}_1\mathbf{k}_o\mathbf{c}
\end{aligned}
\tag{31}
$$

Let the nonlinear system in (7) be augmented with the observer in (30), and define

$$\mathbf{E} \triangleq \left[\xi^T, \tilde{\xi}^T\right]^T, \tag{32}$$

where $\tilde{\xi} = \hat{\xi} - \xi$. Then, the dynamic of the augmented system can be described as

$$\begin{aligned}\dot{\mathbf{E}} &= \underbrace{\begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A} - \mathbf{k}_o\mathbf{c} \end{bmatrix}}_{\mathbf{A}_0} \mathbf{E} + \underbrace{\begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}}_{\mathbf{b}_0} \left(u_L + \mathbf{k}_c\hat{\xi} + \beta\right) \\ &\quad - \underbrace{\begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}}_{\mathbf{b}_1} \beta + \underbrace{\begin{bmatrix} \mathbf{H} \\ -\mathbf{H} \end{bmatrix}}_{\mathbf{H}_1} \Delta_{\eta}\end{aligned} \tag{33}$$

where $\beta \triangleq \Delta - u_{ad} - u_R$.
The available output signal, defined in (9), can be represented as

$$\rho = \gamma \mathbf{k}_c \hat{\xi} = \gamma \left[\mathbf{k}_c \quad \mathbf{k}_c\right] \mathbf{E}. \tag{34}$$

Moreover, using (3) and (32), the following upper bound can be given:

$$\left\|\Delta_{\eta}\right\| \le (c_1 + c_2)\left\|\mathbf{E}\right\|, \tag{35}$$

Lemma 2: *The transfer function* $\gamma\left[\mathbf{k}_c \ \ \mathbf{k}_c\right]\left(s\mathbf{I} - \mathbf{A}_0\right)^{-1}\mathbf{b}_0$ *is Strictly Positive Real (SPR).*

Proof: Let define the positive definite matrix $\mathbf{P} \triangleq \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}$. Then, there exist $\mathbf{Q} = \mathbf{Q}^T > 0$ such that the Lyapunov equation $\mathbf{P}\mathbf{A}_0 + \mathbf{A}_0^T\mathbf{P} = -\mathbf{Q}$ is satisfied. Substituting $\mathbf{A}_0$ from (33) and using (11) and (31) yields

$$\begin{aligned}\mathbf{P}\mathbf{A}_0 + \mathbf{A}_0^T\mathbf{P} &= \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}\begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A} - K_o\mathbf{c} \end{bmatrix} + \begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A} - \mathbf{k}_o\mathbf{c} \end{bmatrix}^T \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix} \\ &= -\underbrace{\begin{bmatrix} \mathbf{Q}_1 & \left(\mathbf{Q}_1 + \mathbf{P}_1\mathbf{k}_o\mathbf{c}\right) \\ \left(\mathbf{Q}_1 + \mathbf{c}^T\mathbf{k}_o^T\mathbf{P}_1\right) & \mathbf{Q}_2 + \left(\mathbf{Q}_1 + \mathbf{c}^T\mathbf{k}_o^T\mathbf{P}_1\right)\mathbf{Q}_1^{-1}\left(\mathbf{Q}_1 + \mathbf{P}_1\mathbf{k}_o\mathbf{c}\right) \end{bmatrix}}_{\mathbf{Q}}\end{aligned} \tag{36}$$

By using the determinant formula

$$\det\begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{C} & \mathbf{B} \end{bmatrix} = \det(\mathbf{A})\det\left[\mathbf{B} - \mathbf{C}\mathbf{A}^{-1}\mathbf{D}\right],$$

it is easy to conclude that **Q** is a positive definite matrix. Moreover,

$$\begin{aligned}\mathbf{P}\mathbf{b}_0 &= \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}\begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_1\mathbf{b} \\ \mathbf{P}_1\mathbf{b} \end{bmatrix} \\ &= \begin{bmatrix} \gamma\mathbf{k}_c^T \\ \gamma\mathbf{k}_c^T \end{bmatrix} = \gamma\left[\mathbf{k}_c \quad \mathbf{k}_c\right]^T\end{aligned}. \tag{37}$$

Hence, according to Leftshetz-Kalman-Yakobuvich (LKY) Lemma, $\gamma\left[\mathbf{k}_c \ \ \mathbf{k}_c\right]\left(s\mathbf{I} - \mathbf{A}_0\right)^{-1}\mathbf{b}_0$ is a SPR transfer function.

□

4 Stability analysis

In this section, it is shown analytically that the error dynamic is ultimately bounded. The analysis is based on the Lyapunov direct method.

Theorem: *Consider the linear controller term* u_L *in (8), the neuro-adaptive control term* u_{ad} *in (19) with the adaptation rules given in (24), and the robustifying control term* u_R *in (27). Then, the error signals* $\mathbf{E}$, $\tilde{\mathbf{w}}$, $\tilde{\mathbf{V}}$ *and* $\tilde{\varphi}$ *in the closed-loop system (33) are uniformly ultimately bounded.*

Proof: Let define the Lyapunov function as

$$L = \frac{1}{2}\left(\hat{\xi}^T \mathbf{P}_1 \hat{\xi} + \tilde{\xi}^T \mathbf{P}_2 \tilde{\xi}\right) + \frac{1}{2\gamma_{\mathbf{w}}}\|\tilde{\mathbf{w}}\|^2 + \frac{1}{2\gamma_V}\|\tilde{\mathbf{V}}\|_F^2 + \frac{1}{2\gamma_\varphi}|\tilde{\varphi}|^2 .$$

Using (32), this Lyapunov function can be represented as

$$L = \frac{1}{2}\mathbf{E}^T \mathbf{P}\mathbf{E} + \frac{1}{2\gamma_{\mathbf{w}}}\|\tilde{\mathbf{w}}\|^2 + \frac{1}{2\gamma_V}\|\tilde{\mathbf{V}}\|_F^2 + \frac{1}{2\gamma_\varphi}|\tilde{\varphi}|^2 \tag{38}$$

where $\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}$. Define $\tilde{\varphi} \triangleq \varphi^* - \varphi$, where φ^* is the ideal gain of its estimated value φ, and $\mathbf{w}^*$ and $\mathbf{V}^*$ are the ideal constant weights defined in (18). Then, from (21) $\dot{\mathbf{w}} = -\dot{\tilde{\mathbf{w}}}$ and $\dot{\mathbf{V}} = -\dot{\tilde{\mathbf{V}}}$. Using (33), the time-derivative of L becomes

$$\begin{aligned}\dot{L} = -\frac{1}{2}\mathbf{E}^{\mathrm{T}}&\left(\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}\begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A} - \mathbf{k}_o\mathbf{c} \end{bmatrix} + \begin{bmatrix} \mathbf{A} - \mathbf{b}\mathbf{k}_c & -\mathbf{b}\mathbf{k}_c \\ 0 & \mathbf{A} - \mathbf{k}_o\mathbf{c} \end{bmatrix}^T \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}\right)\mathbf{E} \\ &+ \mathbf{E}^T \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_1 + \mathbf{P}_2 \end{bmatrix}\mathbf{b}_0\left(\beta + u_L + \mathbf{k}_c\hat{\xi}\right) - \mathbf{E}^T\mathbf{P}\mathbf{b}_1\beta + \mathbf{E}^T\mathbf{P}\mathbf{H}_1\mathbf{\Delta}_{\mathbf{\eta}} - \frac{1}{\gamma_w}\tilde{\mathbf{w}}^T\dot{\mathbf{w}} - \frac{1}{\gamma_V}\mathrm{tr}\left(\tilde{\mathbf{V}}^T\dot{\mathbf{V}}\right) - \frac{1}{\gamma_\varphi}\tilde{\varphi}\dot{\varphi}\end{aligned}$$

Using (34), (36) and (37), it yields

$$\dot{L} = -\frac{1}{2}\mathbf{E}^{\mathrm{T}}\mathbf{Q}\mathbf{E} + \rho\,\beta - \mathbf{E}^T\mathbf{P}\mathbf{b}_1\beta + \mathbf{E}^T\mathbf{P}\mathbf{H}_1\mathbf{\Delta}_{\mathbf{\eta}} - \frac{1}{\gamma_w}\tilde{\mathbf{w}}^T\dot{\mathbf{w}} - \frac{1}{\gamma_V}\mathrm{tr}\left(\tilde{\mathbf{V}}^T\dot{\mathbf{V}}\right) - \frac{1}{\gamma_\varphi}\tilde{\varphi}\dot{\varphi}$$

Substituting $\beta = \tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}}\mathbf{V}^T\boldsymbol{\zeta}\right) + \mathrm{tr}\left(\tilde{\mathbf{V}}^T\boldsymbol{\zeta}\mathbf{w}^T\dot{\boldsymbol{\sigma}}\right) + \delta - u_R$ from (20) and (33), results in

$$\dot{L} = -\frac{1}{2}\mathbf{E}^{\mathrm{T}}\mathbf{Q}\mathbf{E} + \rho\left(\tilde{\mathbf{w}}^T\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}}\mathbf{V}^T\boldsymbol{\zeta}\right) + \mathrm{tr}\left(\tilde{\mathbf{V}}^T\boldsymbol{\zeta}\mathbf{w}^T\dot{\boldsymbol{\sigma}}\right)\right) - \frac{1}{\gamma_w}\tilde{\mathbf{w}}^T\dot{\mathbf{w}} - \frac{1}{\gamma_V}\mathrm{tr}\left(\tilde{\mathbf{V}}^T\dot{\mathbf{V}}\right) + \rho\left(\delta - u_R\right) - \frac{1}{\gamma_\varphi}\tilde{\varphi}\dot{\varphi} - \mathbf{E}^T\mathbf{P}\mathbf{b}_1\beta + \mathbf{E}^T\mathbf{P}\mathbf{H}_1\mathbf{\Delta}_{\mathbf{\eta}}$$

Using the bounds (26) and (35), and the robustifying control term in (27) yields

$$\begin{aligned}\dot{L} \le -\frac{1}{2}Q_m\|\mathbf{E}\|^2 &+ \tilde{\mathbf{w}}^T\left(\rho\left(\boldsymbol{\sigma} - \dot{\boldsymbol{\sigma}}\mathbf{V}^T\boldsymbol{\zeta}\right) - k_w\mathbf{w} - \frac{1}{\gamma_w}\dot{\mathbf{w}}\right) + \mathrm{tr}\left(\tilde{\mathbf{V}}^T\left(\rho\boldsymbol{\zeta}\mathbf{w}^T\dot{\boldsymbol{\sigma}} - k_V\mathbf{V} - \frac{1}{\gamma_V}\dot{\mathbf{V}}\right)\right) + k_w\tilde{\mathbf{w}}^T\mathbf{w} \\ &+ k_V\,\mathrm{tr}\left(\tilde{\mathbf{V}}^T\mathbf{V}\right) + |\rho|\left(\varphi^* - \varphi\right)\chi - \frac{1}{\gamma_\varphi}\tilde{\varphi}\dot{\varphi} + \|\mathbf{E}\|\|\mathbf{P}\mathbf{b}_1\|\beta_M + (c_1 + c_2)\|\mathbf{P}\mathbf{H}_1\|\|\mathbf{E}\|^2,\end{aligned} \tag{39}$$

where from (25) and (29) $\beta_M \triangleq 2mM_w + \varepsilon_{1M} + U_M$ and $Q_m \triangleq \lambda_{\min}(\mathbf{Q})$ denotes the smallest eigenvalue of $\mathbf{Q}$. By applying the adaptation rules in (24), it gives

$$\dot{L} \le -\left(\frac{1}{2}Q_m - (c_1 + c_2)\|\mathbf{P}\mathbf{H}_1\|\right)\|\mathbf{E}\|^2 - k_w\|\tilde{\mathbf{w}}\|^2 + k_wM_w\|\tilde{\mathbf{w}}\| - k_V\|\tilde{\mathbf{V}}\|^2 + k_VM_V\|\tilde{\mathbf{V}}\| + \tilde{\varphi}\left(|\rho|\chi - \frac{1}{\gamma_\varphi}\dot{\varphi}\right) + \beta_M\|\mathbf{P}\mathbf{b}_1\|\|\mathbf{E}\|.$$

Using the learning rules in (28) and completing the square terms yields

$$\dot{L} \le -A_E\|\mathbf{E}\|^2 - \left(k_w - 1\right)\|\tilde{\mathbf{w}}\|^2 - \left(k_V - 1\right)\|\tilde{\mathbf{V}}\|^2 + R, \tag{40}$$

where

$$\left.\begin{aligned} A_E &= \left(\frac{1}{2}Q_m - (c_1 + c_2)\|\mathbf{P}\mathbf{H}_1\| - 1\right) \\ R &= \frac{\left(k_wM_w\right)^2}{4} + \frac{\left(k_VM_V\right)^2}{4} + \frac{\left(\beta_M\|\mathbf{P}\mathbf{b}_1\|\right)^2}{2}\end{aligned}\right\} \tag{41}$$

Now, assume c_1 and c_2 are selected such small that matrix $\mathbf{Q}$ can be find to ensure

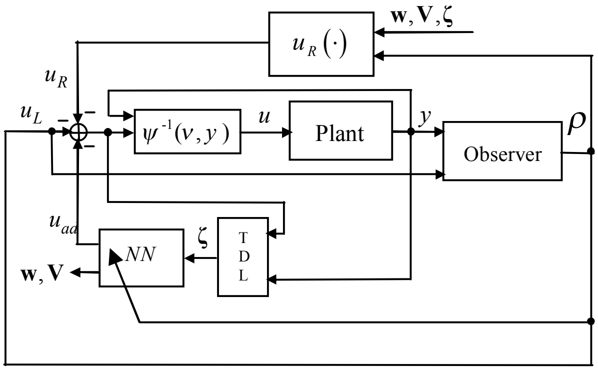

Figure 1 Bock diagram of the proposed control

$$Q_m > 2(c_1 + c_2)\|\mathbf{PH}_1\| + 2, \tag{42}$$

and select $k_w > 1$ and $k_V > 1$, and define the following compact sets around the origin:

$\Omega = \left\{(\mathbf{E}, \tilde{\mathbf{w}}, \tilde{\mathbf{V}}) \middle| A_E \|\mathbf{E}\|^2 + (k_w - 1)\|\tilde{\mathbf{w}}\|^2 + (k_V - 1)\|\tilde{\mathbf{V}}\|^2 \le R\right\}$ Equation (40) shows that $\dot{L} < 0$ once the errors are outside the compact set Ω. Hence, according to the standard Lyapunov theorem extension (Narendra and Annaswamy, 1987; Ge and Zhang, 2003), the error trajectories $\mathbf{E}$, $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{V}}$ are ultimately bounded. □

Figure 1 shows the block diagram of the proposed control method.

Remark 2: *Let define* $\Omega_E \triangleq \left\{\mathbf{E} \middle| \|\mathbf{E}\| \le \sqrt{R/A_E}\right\}$. *Then, from (40) it can be conclude that* $\dot{L}$ *is strictly negative as long as* $\mathbf{E}$ *is outside the compact set* Ω_E. *Therefore, there exists a constant time T such that for* $t > T$, *the error* $\mathbf{E}$ *converge to* Ω_E *(Lewis, et al., 1996; Ge and Zhang, 2003). This means that* $\|\boldsymbol{\xi}\| \le \sqrt{R/A_E} = \varepsilon_E$ *and consequently* $\|\mathbf{z}\| \le \varepsilon_E$ *and* $\|\boldsymbol{\eta}\| \le \varepsilon_E$.

Remark 3: *As Eq. (41) shows, the NN reconstruction error embodied in the constant* β_M *increases the error bound. Since* u_{ad} *and* u_R *are designed to approximate* Δ, *the upper bound* $|\Delta - u_{ad} - u_R| \le \beta_M$ *defined in (39) is very conservative, and in practice the real bound would be much smaller. Moreover, the stability result presented here is semi global in the sense that it is local with respect to the approximation domain of* Δ *with* $u_{ad} + u_R$ *and domain of unmatched uncertainty bound defined in (3).*

Remark 4: *It will be shown in the followings that conditions (42) can be satisfied easier by increasing the rate of convergence of the error dynamic. Let define new state variables as* $\boldsymbol{\varsigma} = \mathbf{S}^{-1}\mathbf{E}$ *where* $\mathbf{S}$ *is defined such* $\boldsymbol{\Lambda}_0 = \mathbf{S}^{-1}\mathbf{A}_0\mathbf{S}$ *represents a diagonal matrix. Replacing* $\mathbf{E}$ *by* $\boldsymbol{\varsigma}$ *in (33) and repeating the proof of the Theorem, results in a similar condition as in (42)*

$$\lambda_{\min}(\mathbf{Q}') > 2(c_1 + c_2)\|\mathbf{P}'\mathbf{H}_1'\| + 2 \tag{43}$$

where $\mathbf{Q}' = \mathbf{S}^T\mathbf{QS}$, $\mathbf{P}' = \mathbf{S}^T\mathbf{PS}$ *and* $\mathbf{H}_1' = \mathbf{S}^{-1}\mathbf{H}_1$.
From (7) and (33), $\|\mathbf{H}_1\| = \sqrt{\lambda_{\max}(\mathbf{H}_1^T\mathbf{H}_1)} = 1$ *and* $\mathbf{P}' = \mathbf{P}'^T$. *Hence,*

$$\|\mathbf{P}'\mathbf{H}_1'\| \le \|\mathbf{P}'\|\|\mathbf{S}^{-1}\|\|\mathbf{H}_1\| = \lambda_{\max}(\mathbf{P}')\|\mathbf{S}^{-1}\|. \tag{44}$$

On the other hand, one can conclude that $\lambda_{\max}(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_0^T) < 0$. *Therefore, (Lancaster, 1970),*

$$\lambda_{\max}(\mathbf{P}') \le \frac{\lambda_{\max}(\mathbf{Q}')}{\left|\lambda_{\max}(\boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_0^T)\right|}. \tag{45}$$

Substituting (44) and (45) into (43), it can be represented in a new form as

$$\lambda_{\min}(\mathbf{Q}') > \frac{2(c_1+c_2)\left\|\mathbf{S}^{-1}\right\|\lambda_{\max}(\mathbf{Q}')}{\left|\lambda_{\max}(\mathbf{\Lambda}_0+\mathbf{\Lambda}_0^T)\right|}+2 \qquad (46)$$

As Equation (46) implies, the necessary condition given in (42) can be satisfied by accelerating the convergence rate of the error dynamics (i.e. by achieving larger eigenvalues for $\mathbf{A}_0$ *or equivalently* $\mathbf{\Lambda}_0$*). This can be done by selecting suitable values for vector gains* $\mathbf{k}_c$ *and* $\mathbf{k}_o$ *.*

Remark 5: *When a discontinuous control signal is applied to a system, a phenomenon called chattering appears. Many methods have been proposed in literatures to reduce chattering including continuous approximation of the discontinuous control signal. A continuous approximation of* $\mathrm{sgn}(\rho)$ *in (27) is* $\tanh\left(\rho/\varepsilon\right)$ *or alternatively* $\frac{\rho}{|\rho|+\delta}$ *where* $\varepsilon>0$ *and* $0<\delta<1$*. However, using this continuous function can increase the ultimate error bound, which is proportional to the value of* ε *and* δ *.*

The block diagram of the proposed method is shown in Figure 1.

5 Simulation examples

5.1 Flexible link manipulator

The proposed controller is applied to stabilize the Flexible Link Manipulator (FLM) system depicted in Figure 2 (Wang and Vidiasagar, 1991). Points on the beam have their position fixed by the variable *x*, which is the distance of that point from the hub of the motor driving the beam. The elastic deformation at *x* is given as $a(x,t)$ and $y(x,t)$ is the net movement of that point. In other words

$$y(x,t)=\theta(t)+\frac{a(x,t)}{x}$$

where θ is the angular rotation of the beam and $a(x,t)=\sum_{i=1}^{n}q_i(t)\phi_i(x)$ in which $\phi_i(x)$ are the clamped-free eigenfunctions, and *n* is the number of considered resonance frequencies.
This system can be described by the following equations when only the first mode of the resonance frequency is considered, i.e. $n=1$ (Wang and Vidiasagar, 1991):

$$\begin{cases}\ddot{\theta}=\frac{\left(1-\omega_1^2\right)}{\kappa}q_1+\frac{1}{a}q_1\dot{\theta}^2\\ \ddot{q}_1=\frac{q_1}{\kappa}\left(q_1^2+I_b+I_h\right)\left(\left(\omega_1^2-1\right)-\dot{\theta}^2\right)-\frac{2}{\kappa}q_1\dot{q}_1\dot{\theta}+\frac{1}{\kappa}\tau\end{cases}$$

where $\theta, \omega_1, I_b, I_h$, and τ are the deflection, the first resonance frequency, the beam moment of inertia, the inertia of motor hub, and the input torque, respectively.

Moreover, $\kappa \triangleq \rho_L\int_0^h\phi_1(x)x\,dx$, where ρ_L and *h* are the mass per unit length and the beam length, respectively. If the tip position of the beam is considered as the output, i.e. $y=\theta+\frac{a(h,t)}{h}$, then, the zero dynamics corresponding to this output are unstable. Hence, a new output is defined as $y_n=\theta+p\frac{a(h,t)}{h}$, with $-1<p<1$, where *p* is selected such that the zero dynamics are stable (Talebi and Patel, 2005; Madhavan and Singh, 1991). This new output variable is physically realizable, as sensors can be designed to measure θ and $a(h,t)$ separately. Since the proposed method can also be applied to non-minimum phase systems, the actual output (tip of the beam) is selected here as the measurable output. Let define the new state variables as

$$z_1=y=\theta+\frac{\phi_1(h)}{h}q_1,\; z_2=\dot{\theta}+\frac{\phi_1(h)}{h}\dot{q}_1, \eta_1=\theta, \eta_2=\dot{\theta}.$$

Then,

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = (z_1-\eta_1)\left(\left(\frac{h}{\phi(h)}\right)^2 (z_1-\eta_1)^2 + I_b + I_h\right)\left(\frac{(\omega_1^2-1)}{\kappa} - \frac{1}{\kappa}\eta_2^2\right) \\ + \frac{h}{\kappa\phi(h)}(z_1-\eta_1)\left(\eta_2^2 - 2(z_2-\eta_2)\eta_2 + (1-\omega_1^2)\right) + \frac{\phi(h)}{\kappa h}u \\ \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = \frac{h(1-\omega_1^2)}{\kappa\phi(h)}(z_1-\eta_1) + \frac{h}{\kappa\phi(h)}(z_1-\eta_1)\eta_2^2 \end{cases}$$

It is straightforward to check that zero dynamics are not asymptotically stable when $\omega_1 > 1$, yielding a non-minimum phase system.
Assume the following linear model of the system is available in presence of parameters uncertainties:

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = \frac{(1-\hat{\omega}_1^2)}{\kappa}\left(\frac{h}{\phi(h)} - (\hat{I}_b + \hat{I}_h)\right)(z_1-\eta_1) + u_L \\ \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = \frac{h(1-\hat{\omega}_1^2)}{\kappa\phi(h)} z_1 - \frac{h(1-\hat{\omega}_1^2)}{\kappa\phi(h)}\eta_1 \end{cases}$$

where $\hat{I}_h, \hat{I}_b$ and $\hat{\omega}_1$ are the estimates of their corresponding parameters I_h, I_b and ω_1, respectively. Consider the best available approximation of model uncertainties defined in (4), as $\hat{\psi}(u,y) = cu$ where c is properly selected constant to satisfy Assumption 2; that is

$$c > 0.5\frac{\phi(h)}{\kappa h}.$$

The NN is of MLP type with 10 neurons in one hidden layer and tangent hyperbolic activation function. The weights are initialised randomly using small numbers. The input vector applied to the NN for $n_1 = 4 \geq n$ is

$$\zeta = [y(t), y(t-T_d), y(t-2T_d), y(t-3T_d), u(t-T_d), u(t-2T_d)]^T$$

with $T_d = 10$ msec. In addition, learning coefficients are selected as $\gamma_W = \gamma_V = 0.003$, $k_w = k_V = 1.2$, and

$$\mathbf{k}_c = [35.47, 9.44, -36.89, -13.59],\ \mathbf{k}_o = [47, 782.5, -900, -3046].$$

The simulations are carried out using the following parameters and initial conditions:

$$h = 1.2 \text{ m},\quad \rho_L = 2 \text{ kg/m},\quad I_b = 1.15 \text{ kgm}^2,$$
$$I_b = 0.8 \text{ kgm}^2,\quad \omega_1 = 3 \text{ rad/sec}.$$
$$z_1(0) = \eta_1(0) = 1.0 \text{ rad},\quad z_1(0) = \eta_1(0) = 0 \text{ rad/sec}.$$

Simulation results are depicted in Figures 3-5. First, performance of the closed-loop system is evaluated without parameters uncertainties, namely $\hat{I}_b = I_b$, $\hat{I}_h = I_h$, and $\hat{\omega}_1 = \omega_1$. Figure 3 shows response of the closed-loop system. Observe that both the proposed and the linear optimal controller can stabilize the system.

Next, simulation is repeated in presence of parameter uncertainties with $\hat{I}_b = 1.2 I_b$ $\hat{I}_h = 1.2 I_h$, and $\hat{\omega} = 1.2\omega$. Figure 4 shows that the proposed method can stabilize state variables while the linear controller cannot stabilize the system. As Figure 5a shows, the proposed approach cancels out uncertainties, adaptively with relatively good accuracy. Moreover, Figure 5 shows validity of unmatched uncertainties bound in the operating region of the system defined in (3), boundedness of the estimation error, and the adaptive weights.

5.2 *Translational oscillator rotational actuator*

For the sake of comparison with other advanced control methods reported in literatures, the proposed controller is also applied to stabilize the TORA system (see Figure 6), which is described by Karagiannis et al. (2005). The dynamic equations of the system are

$$\begin{cases} (M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = -kx \\ (J+ml^2)\ddot{\theta} + ml\cos\theta\,\ddot{x} = \tau, \end{cases}$$

where θ is the angle of rotation, x is the translational displacement, and τ is the control torque. The positive constants k, l, J, M, and m denote the spring stiffness, the radius of rotation, the moment of inertia, the mass of the cart, and the eccentric mass, respectively. When the angle of rotation θ is defined as the output, the system is a weakly minimum phase.

Figure 7 compares the simulation results of the proposed method with the remarkable back-stepping approach (Karagiannis et al., 2005). All the parameter values and the initial conditions are the same for both methods. It should be mentioned that design of the back- stepping controller is complicated and requires some guess work. Moreover, the application of this method is restricted only to systems, where the nonlinearities depend only on the system output. On the other hand, the proposed method can be applied to plants that are more general, and only a linear approximation of the system is required to design the controller. In addition, the design procedure is relatively straightforward. However, the main advantage of the back-stepping approach is that it guarantees global stability of the system while the proposed control method in this paper can guarantee semi global stability (see Remark 3).

The simulations are carried out using a MLP with five neurons in the hidden layer with tangent hyperbolic activation functions. The input vector to the NN for $n_1 = 4 \geq n$ is

$$\zeta = [y(t), y(t-T_d), y(t-2T_d), y(t-3T_d), u(t-T_d), u(t-2T_d)]^T$$

with $T_d = 10$ m sec. The learning constants are selected as $\gamma_{\mathbf{w}} = \gamma_{\mathbf{V}} = 0.2$, $\gamma_\varphi = 1$ and $k_w = k_V = 5$. The controller and the observer gains are

$$\mathbf{k}_c = [2.58.47, 1, \text{-}403.8, 5.48], \ \mathbf{k}_o = [82, 2746, 25.6, -65.7].$$

As Figure 7 shows, the response of the closed-loop system using the proposed controller is nearly the same as the response of the back-stepping control method. The unmatched uncertainty cancellation, the norm of the adaptive weights, and the estimation errors are shown in Figure 8.

6 Conclusions

A direct adaptive output-feedback stabilization method for nonlinear non-minimum phase systems was proposed in this paper. The proposed method relies on state estimation. The approach can be applied to uncertain non-affine nonlinear systems, from which a linear approximation can be derived. The ultimate boundedness of all states including the internal dynamics and the NN weights was shown analytically using the Lyapunov direct method. Simulation results, performed on a single flexible link and the TORA system showed good performance as compared to the linear optimal controller and the back-stepping control method, respectively.

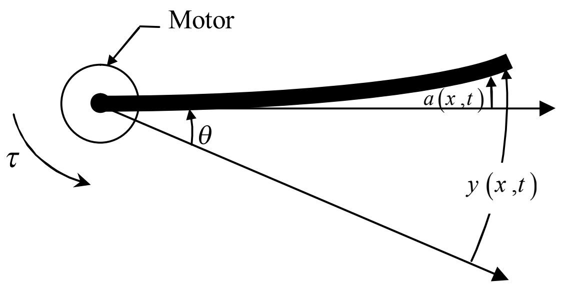

Figure 2 Single flexible link manipulator

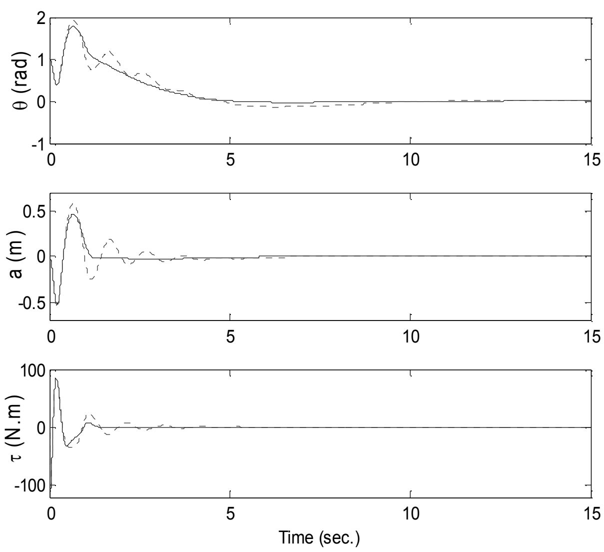

Figure 3 Response of the FLM system without parameter uncertainty. Dotted line: linear optimal controller, solid line: the proposed controller

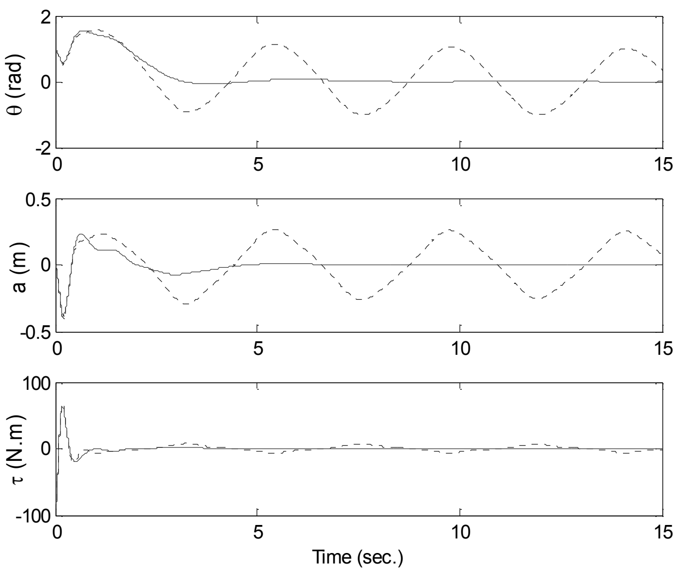

Figure 4 Response of the FLM system with parameter uncertainty.
Dotted line: the linear optimal controller, solid line: the proposed controller

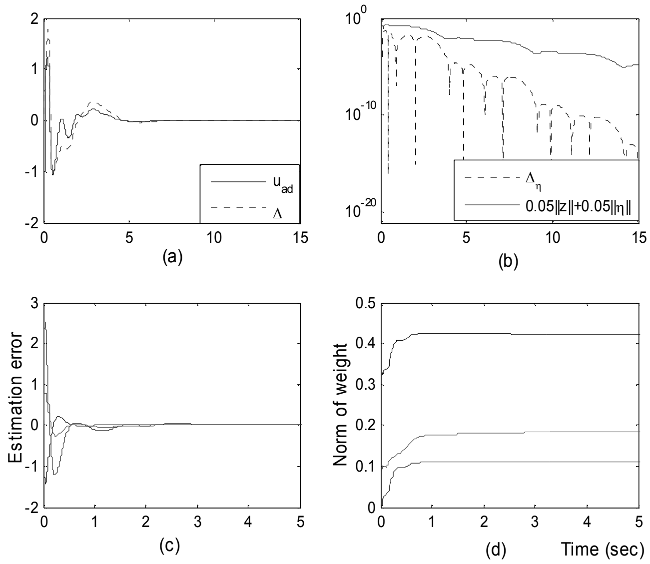

Figure 5 Response of the FLM system with parameter uncertainty. (a) matched uncertainty cancellation, (b) validation of unmatched uncertainties bound, (c) errors of estimation $\tilde{\boldsymbol{\xi}}$, (d) normalized norm of adaptive weights $\mathbf{w}, \mathbf{V}$ and φ

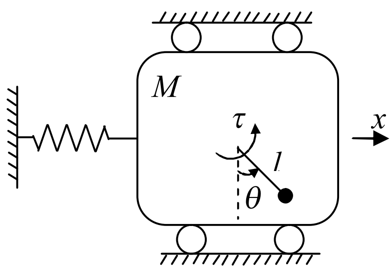

Figure 6 A translational oscillator rotational actuator

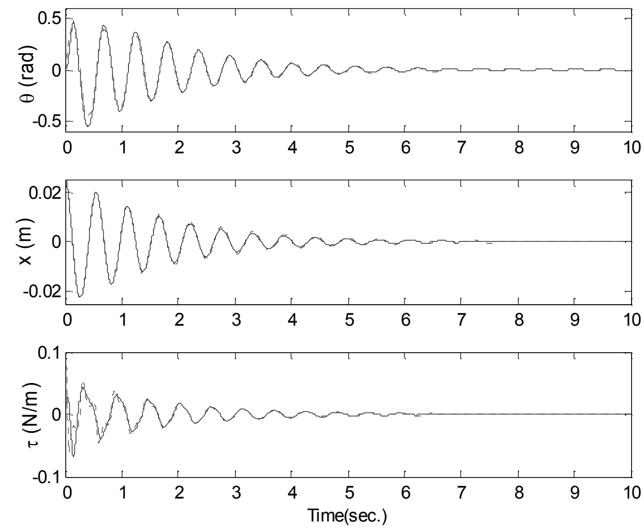

Figure 7 Response of the TORA system.
Dotted line: back-stepping based controller, solid line: the proposed controller

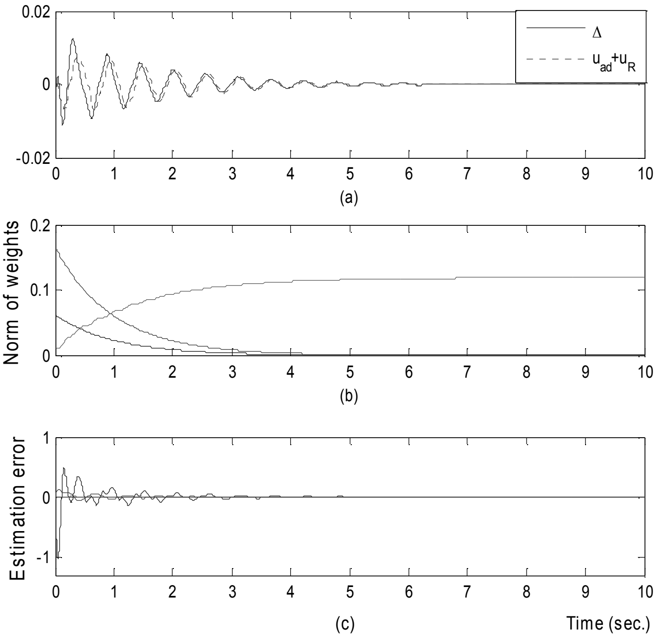

Figure 8 Response of the TORA system:

(a) matched uncertainty cancellation (b) norm of weights (c) errors of estimation $\tilde{\xi}$.

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