Parameter Selection Strategies in Finite Element Model Updating

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In FE model updating, as in any identification procedure, we select some parameters in the model, and try to fine-tune them to minimize the discrepancy between the model predictions and the measured data. The paper compares the performance of the generic element matrices, recently introduced by the authors, with other selection strategies for finite element model updating. The updated models obtained from these methods are compared with the measured data and rated according to their ability to produce the measured data within and beyond the frequency range used in the updating, and more importantly, according to their ability to predict the effect of design changes.

1 Introduction

Updating procedures are concerned with the reconstruction of a dynamical model, near to the finite element model of a structure, which predicts the measured response of the structure. In model updating we have a system, and we have modeled it, but our predictions do not agree with experiment. In effect what we want to do is to change the model, a little, so that it will *model the behavior of the system.* The updating is thus a *system (or model) identification problem:* the problem is not an identification problem in the widest sense; we have a model which predicts results near, in some sense, to experimental values; we just need to fine-tune it. Here we are concerned exclusively with small undamped vibration of a conservative system; such systems form the basis, i.e., the starting point, for most research in the field.

One of the fundamental questions in updating relates to the criteria for allowable changes in the initial mass and stiffness matrices. In early papers, for example Baruch and Bar Itzhack (1978), Berman (1979), Berman and Nagy (1983), Baruch (1984), any symmetric changes in the matrices were allowed. This may give a model consistent with the test data, but the updated matrices are fully populated and the structure of the updated model may not mirror the way the elements are connected together. This may not matter if the original matrices are densely populated, but will matter if they are sparse. To keep the resemblance between the structure of the model and the structure of the object, other researchers, Kabe (1985), Caesar and Peter (1987), Kammer (1988), Smith and Beattie (1991), retained the pattern of zeros of the FE model and allowed symmetric changes only in the non-zero entries of the model matrices. Caesar et al. and Smith et al. have also paid attention to the conflicting requirements of definiteness and sparsity, particularly for inconsistent data.

A third approach exemplified by Hall, Calkins and Sholar (1970), Tlusty and Ismail (1980), Chen and Graba (1980), Hoff and Natke (1989), Lim (1990), O'Callahan (1990) is to adjust the physical parameters, e.g., the flexural rigidities, masses or stiffnesses, and keep the (numerical) matrices that arise from integration of assumed element shape functions unchanged. As long as the physical parameters remain positive, the model will have the correct definiteness properties; by the way in which it is constructed, it has the same connectivity properties as the original model. In practice however, this class of models is too restrictive, and performs badly, especially for data which is inconsistent with the assumed model, as is always the case in practice. If the initial FE model neglects some *effects,* such as the shear deformation or coupling between bending and twisting modes, then the method will be unable to correct the FE model.

The authors have recently proposed a fourth approach, in which they introduce *generic* element mass and stiffness matrices. The essence of the idea may be presented by considering the various element mass and stiffness matrices which has been proposed for the simplest structural entity, the rod element.

Consider a straight, thin rod of length L in longitudinal vibration, having density ρ, cross-section A, and Young's modulus E. There is for example:

(a) a lumped mass matrix, and a stiffness matrix based on assumed modes $N_1 = x/L$, $N_2 = 1 - x/L$

$$\mathbf{M}^e = m_0 \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \mathbf{K}^e = k_0 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

where $m_0 = \rho AL$, $k_0 = EA/L$.

(b) a consistent pair based on $N_1 = x/L$, $N_2 = 1 - x/L$

$$\mathbf{M}^e = \frac{m_0}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{K}^e = k_0 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

(c) a consistent pair based on $N_1 = \cos^2 \theta$, $N_2 = \sin^2 \theta$ where $\theta = \pi x/2L$,

$$\mathbf{M}^e = \frac{m_0}{8} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{K}^e = k_0 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

We may form other mass and stiffness matrices by choosing other assumed modes. But we can go even further and concentrate our attention on the matrices $\mathbf{M}^e$, $\mathbf{K}^e$, and not suppose that they arise from some assumed modes. What criteria must such *generic* element matrices for the rod satisfy? $\mathbf{M}^e$ must be positive-definite and $\mathbf{K}^e$ positive semi-definite; $\mathbf{K}^e$ must have rigid body mode [1, 1]; $\mathbf{M}^e$, $\mathbf{K}^e$ must reflect the fact that the rod element is symmetric about its mid-point, so that physically turning it around has no effect. We show in Ahmadian, Gladwell, and Ismail (1994) that the generic mass and stiffness

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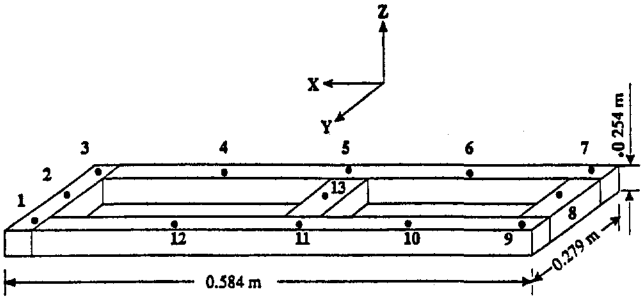

Fig. 1 Frame structure and measured coordinates

matrices for the rod may be satisfied by a single parameter S_2 (in the notation of that paper) and that

$$\mathbf{M}^e = \begin{bmatrix} m_1 & m_{12} \\ m_{12} & m_1 \end{bmatrix}, \quad \mathbf{K}^e = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

where

$$m_1 = \frac{m_0}{4}(1 + S_2^2), \quad m_{12} = \frac{m_0}{4}(1 - S_2^2), \quad k = k_0\gamma S_2^2/4.$$

Here γ is related to the single non-zero eigenvalue of the equation

$$(\mathbf{K}^e - \lambda\mathbf{M}^e)\boldsymbol{\phi} = 0$$

by $\lambda = k_0\gamma/m_0$. The values of S_2 corresponding to the case (a), (b), (c) are 1, $1/\sqrt{3}$, $1/\sqrt{2}$ respectively.

In order to find a model of an extended rod to fit some behavioral data we divide it into generic elements and then use one of the many existing methods to choose the best values of the parameters to fit the data; we may assume that each element has its own value of m_0, k_0, S_2, and γ or that some or all elements have the same values of the parameters. This is the essence of the idea of generic elements. To generalize it we proceed as follows.

A generic element is an object, a structural entity, whose motion is defined by a certain number, r, of degrees of freedom. It will have a mass matrix $\mathbf{M}^e$, symmetric and positive definite and of order r. It will have a certain number, d (≤ 6), rigid body modes, and will have a stiffness matrix $\mathbf{K}^e$, symmetric and positive semi-definite and of rank $r - d$. If the element possesses certain symmetry properties, then its matrices will exhibit these properties. *This appears to be the sum total of requirements for the generic element matrices.* This means that any particular matrices derived by considering certain effects (e.g. shear, torsion and bending coupling, etc) in an element must fall into the general classes described above. However, in the updating procedure, we do not know all the *effects* which must be considered; there may even be some which have not been described in the existing literature. Therefore, instead of trying to incorporate all the possible effects into the matrices by introducing various physical parameters and finding the appropriate values of these parameters, we merely assume that the matrices for the elements belong to the generic families appropriate to these elements, and find the parameters needed to specify members in these families. In this way we can take account not only of those effects we are aware of, but also all the possible effects which can be accommodated by the matrices in the generic family.

In broad terms, model updating has two parts: choosing the set of models among which the model is to be sought; choosing the parameter values specifying the model. This paper is concerned largely with the first part. The set of models based on generic elements is wide; in fact it is the widest possible set based on the chosen degrees of freedom. On the other hand, the set of models provides global mass and stiffness matrices which satisfy the required positivity and connectivity conditions; these are two of the requirements which have plagued updating methods for many years.

In this paper we carry out a comparison between these four different strategies for selecting the updating parameters, by updating the FE model of a frame structure. The remainder of the paper is organized as follows. Section 2 provides a brief discussion of the formulation of *generic element models* for model updating. In Section 3 we compare the predictions of the initial FE model with the test results, and discuss the preprocessing of the test data. Finally in Section 4 we update the frame structure model using different methods and comment on the ability of each method to produce the test results, to predict the behavior of the system outside the frequency range used in the test, and to predict the effects of modifying the structure.

2 Generic Element Models

This section summarizes the analysis of Gladwell and Ahmadian (1996) and generalizes the results given in Section 1 for the rod element. We suppose that the element has r degrees of freedom; its free vibration is governed by

$$(\mathbf{K}^e - \lambda_i\mathbf{M}^e)\boldsymbol{\phi}_i = \mathbf{0}, \quad i = 1, 2, \ldots, r. \tag{1}$$

If the element has $d \leq 6$ rigid-body modes $(\boldsymbol{\phi}_i)_1^d$, and $r - d$ strain modes, we write the $r \times r$ matrix $\boldsymbol{\Phi}^e$ as

$$\boldsymbol{\Phi}^e = [\phi_1, \phi_2, \ldots \phi_d | \phi_{d+1}, \ldots \phi_r] = [\boldsymbol{\Phi}_R, \boldsymbol{\Phi}_S].$$

where R and S denotes *rigid-body* and *strain* respectively. If the modes are normalized w.r.t. $\mathbf{M}^e$, then

$$\boldsymbol{\Phi}^{eT}\mathbf{M}^e\boldsymbol{\Phi}^e = \mathbf{I}, \quad \boldsymbol{\Phi}^{eT}\mathbf{K}^e\boldsymbol{\Phi}^e = \boldsymbol{\Lambda} \tag{2}$$

where $\boldsymbol{\Lambda}^e = [0, 0, \ldots 0, \lambda_{d+1}, \ldots \lambda_r] = [\mathbf{0}, \boldsymbol{\Lambda}_S]$, and

$$\mathbf{K}^e\boldsymbol{\phi}_i = \mathbf{0}, i = 1, 2, \ldots d, \quad \text{i.e.} \quad \mathbf{K}^e\boldsymbol{\Phi}_R = \mathbf{0}. \tag{3}$$

If we knew $\mathbf{M}^e$, $\mathbf{K}^e$ then we could find $\boldsymbol{\Phi}^e$ and $\boldsymbol{\Lambda}^e$. Thus for the simple lumped mass rod model (a) we have

$$\boldsymbol{\Phi}^e = \frac{1}{\sqrt{m_0}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \boldsymbol{\Lambda}^e = \begin{bmatrix} 0 & 0 \\ 0 & 2k_0/m_0 \end{bmatrix}.$$

But we can reverse this procedure: if we know, or specify $\boldsymbol{\Phi}^e$ and $\boldsymbol{\Lambda}^e$ then we can find $\mathbf{M}^e$ and $\mathbf{K}^e$ from Eq. (2), thus

$$\mathbf{M}^e = \boldsymbol{\Phi}^{e-T}\boldsymbol{\Phi}^{e-1}, \quad \mathbf{K}^e = \boldsymbol{\Phi}^{e-T}\boldsymbol{\Lambda}^e\boldsymbol{\Phi}^{e-1}. \tag{4}$$

Note that we are still at the conceptual stage of model formation; $\boldsymbol{\Phi}^e$ has nothing to do with observed models, but is simply a matrix of free vibration modes of a proposed generic element.

The choice of $\boldsymbol{\Phi}^e$ and $\boldsymbol{\Lambda}^e$ for generic elements must conform to Eqs. (2), (3). Thus if we are to form a generic element with r degrees of freedom and d rigid body modes, then we must choose d rigid-body modes $(\phi_i)_1^d$, the positive eigenvalues $(\lambda_i)_{d+1}^r$ and the corresponding eigen-modes $(\phi_i)_{d+1}^r$. Provided

Table 1 Computed and measured natural frequencies

Mode No.	Natural frequency (HZ) FE model	Measured	error %
1	255.8	226.8	12.8
2	277.5	275.2	0.9
3	581.3	537.4	8.3
4	911.3	861.5	6.0
5	1049.4	974.8	8.0

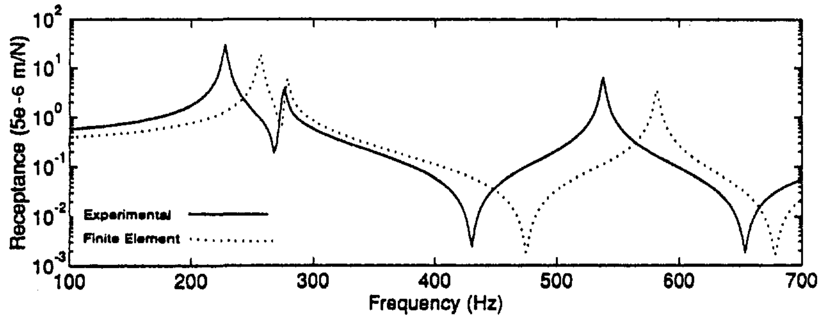

Fig. 2 The measured and predicted FRF (6Z/6Z)

that $\mathbf{\Phi}^e$ is non-singular, the $\mathbf{M}^e$ and $\mathbf{K}^e$ constructed from (4) will have the required form, i.e. $\mathbf{M}^e$ will be positive definite and $\mathbf{K}^e$ will be positive semi-definite with rigid body modes $(\phi_i)_1^d$. We will not actually use the matrix inversion shown in (4) to construct $\mathbf{M}^e$ and $\mathbf{K}^e$, as we will soon show.

We are concerned with updating a given model, not with constructing a model *ab initio.* We will therefore assume that we have a model with element matrices $\mathbf{M}_0^e$, $\mathbf{K}_0^e$ and corresponding modes and eigenvalues $\mathbf{\Phi}_0^e$ and $\mathbf{\Lambda}_0^e$, which we wish to update. We will do this by relating the new modes, $\mathbf{\Phi}^e$, which we will choose, to the existing ones $\mathbf{\Phi}_0^e$, i.e.

$$\mathbf{\Phi}_0^e = \mathbf{\Phi}^e\mathbf{S}, \tag{5}$$

where S is some non-singular matrix. We will also have to choose the updated strain eigenvalues forming the diagonal matrix $\mathbf{\Lambda}_S$.

The correspondence (5) is very general -too general for practical updating- and we therefore restrict it. On physical grounds we now restrict $\mathbf{S}$ so that the number of rigid body modes remains the same, d, and the new rigid body modes are linear combinations of the original ones. Equation (5) now becomes

$$[\mathbf{\Phi}_{0R}, \mathbf{\Phi}_{0S}] = [\mathbf{\Phi}_R, \mathbf{\Phi}_S]\begin{bmatrix}\mathbf{S}_R & \mathbf{S}_{RS}\\ \mathbf{0} & \mathbf{S}_S\end{bmatrix}, \tag{6}$$

i.e.

$$\mathbf{\Phi}_{0R} = \mathbf{\Phi}_R\mathbf{S}_R, \quad \mathbf{\Phi}_{0S} = \mathbf{\Phi}_R\mathbf{S}_{RS} + \mathbf{\Phi}_S\mathbf{S}_S.$$

This means that the new strain modes may be combinations of *all* the original modes. Note that this will not interfere with the orthogonality of the new modes with respect to the new mass and stiffness matrices, because the new matrices $\mathbf{M}^e$, $\mathbf{K}^e$ will be formed from Eq. (4), which automatically imply the orthogonality relations (2).

We may further restrict $\mathbf{S}$ on symmetry considerations. If the initial model is governed by some symmetry group, then its modes $\mathbf{\Phi}_0^e$ will reflect the properties of the group. If the new, generic model retains this symmetry, then new modes in $\mathbf{\Phi}^e$ with a particular symmetry will be linear combinations of the old modes with the same symmetry; this will produce diagonal blocks in $\mathbf{S}$. In particular, if the new and old models have the same center of mass and principal axes at the element level, then $\mathbf{S}_R$ will be diagonal, i.e., rigid body modes will remain the same, apart from a constant; if they have the same mass and principal moments of inertia, then $\mathbf{S}_R$ will be the unit matrix $\mathbf{I}_d$.

Inserting Eq. (5) into (4) we find

$$\mathbf{M}^e = \mathbf{\Phi}_0^{e-T}\mathbf{S}^T\mathbf{S}\mathbf{\Phi}_0^{e-1}, \quad \mathbf{K}^e = \mathbf{\Phi}_0^{e-T}\mathbf{S}^T\mathbf{\Lambda}^e\mathbf{S}\mathbf{\Phi}_0^{e-1}.$$

Now using the fact that $\mathbf{\Phi}_0^{e-1} = \mathbf{\Phi}_0^e\mathbf{M}_0^e$ we find

$$\mathbf{M}^e = \mathbf{M}_0^e\mathbf{\Phi}_0^e\mathcal{M}\mathbf{\Phi}_0^{eT}\mathbf{M}_0^e, \tag{7}$$

$$\mathbf{K}^e = \mathbf{M}_0^e\mathbf{\Phi}_{0S}\mathcal{K}\mathbf{\Phi}_{0S}^T\mathbf{M}_0^e, \tag{8}$$

where

$$\mathcal{M} = \mathbf{S}^T\mathbf{S}, \quad \mathcal{K} = \mathbf{S}_S^T\mathbf{\Lambda}_S\mathbf{S}_S.$$

Equations (7) and (8) show the terms which may be updated appear in two symmetric positive definite matrices $\mathcal{M}$ and $\mathcal{K}$ of orders r and $r-d$ respectively. These equations relate the mass and stiffness matrices of the generic element to those of the original model.

To set up the generic elements in the approach described above, we started with the eigenvalue problem for the element; we coupled $\mathbf{K}^e$ and $\mathbf{M}^e$. As an alternative, we may consider $\mathbf{K}^e$ and $\mathbf{M}^e$ separately, writing

Table 2 Natural frequencies of the updated models after convergence (Hz)

Mode No.	Predicted by FEM	Baruch's Method	Changing non-zeros	Changing non-zeros + rigid modes	Measured
Rigid	0	0	1197.6 i	0	0
body	0	0	456.8 i	0	0
modes	0	0	206.7	0	0
1	255.8	226.8	226.8	226.8	226.8
2	277.5	275.2	275.2	275.2	275.2
3	581.3	537.4	537.4	537.4	537.4
4	911.3	911.3	548.3	862.6	861.5
5	1049.4	1049.4	652.0	897.6	974.8

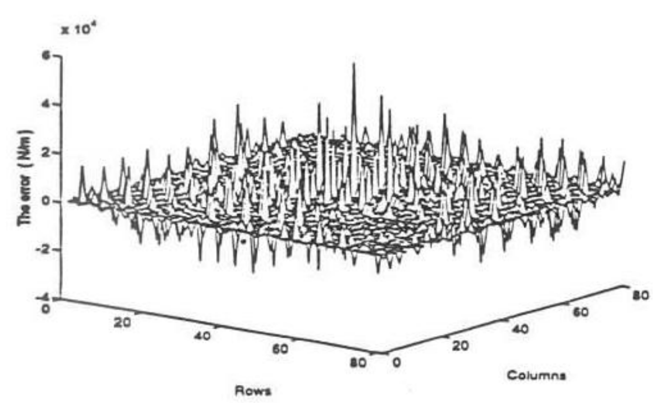

Fig. 3 The difference matrix K_0-K_{Baruch} (modes 1–3)

Table 3 Natural frequencies of the updated models after convergence (Hz)

Mode No.	Predicted by FEM	Physical Parameter	Nobari's Method	Measured
Rigid	0	0	102.5 i	0
body	0	0	0	0
modes	0	0	67.6	0
1	255.8	255.6	255.5	226.8
2	277.5	277.4	277.0	275.2
3	581.3	580.1	580.5	537.4
4	911.3	911.6	905.1	861.5
5	1049.4	1043.2	1040.6	974.8

$$\mathbf{K}^e = \mathbf{U}\Lambda\mathbf{U}^T = \sum_{i=d+1}^{r} \lambda_i \mathbf{u}_i \mathbf{u}_i^T, \tag{9}$$

$$\mathbf{M}^e = \mathbf{V}\Sigma\mathbf{V}^T = \sum_{i=1}^{r} \sigma_i \mathbf{v}_i \mathbf{v}_i^T, \tag{10}$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and

$$\mathbf{U}^T\Phi_R = 0. \tag{11}$$

We may thus define a family by starting from some original model $\mathbf{K}_0^e$ and $\mathbf{M}_0^e$ and defining the new $\mathbf{U}$ and $\mathbf{V}$ matrices by

$$\mathbf{U} = \mathbf{U}_0\mathbf{R}, \quad \mathbf{V} = \mathbf{V}_0\mathbf{T}, \tag{12}$$

where $\mathbf{R}$, $\mathbf{T}$ are orthogonal matrices, and, if need be, by specifying new values of $\Lambda = (\lambda_i)_{d+1}^r$, $\Sigma = (\sigma_i)_1^r$. Thus

$$\mathbf{K}^e = \mathbf{U}_0\mathbf{R}\Lambda\mathbf{R}^T\mathbf{U}_0^T, \tag{13}$$

$$\mathbf{M}^e = \mathbf{V}_0\mathbf{T}\Sigma\mathbf{T}^T\mathbf{V}_0^T. \tag{14}$$

Again, when the element belongs to some symmetry group, the eigenvectors $\mathbf{u}_i$, $\mathbf{v}_i$ reflect the symmetry properties of the group, and $\mathbf{R}$, $\mathbf{T}$ are made up of diagonal blocks.

To reduce the number of unknowns in the symmetric products $\mathbf{R}\Lambda\mathbf{R}^T$ and $\mathbf{T}\Sigma\mathbf{T}^T$, we can update only the dominant modes of each matrix and keep the remainder unchanged. Ross (1971) has suggested that in modeling a dynamic system, with $\mathbf{K}$ and $\mathbf{M}$, it is important to ensure that $\mathbf{M}$ and the flexibility matrix $\mathbf{F}$ are modeled correctly. Using Ross's concept at the element level, we therefore must correctly model the higher modes of $\mathbf{M}^e$ and the *lower* modes of $\mathbf{K}^e$.

Table 4 Natural frequencies of the updated models after convergence (Hz)

Mode No.	Predicted by FEM	LS Solution	Adjusted Solution	Measured
Rigid	0	0	0	0
body	0	0	0	0
modes	0	0	0	0
1	255.8	226.8	226.8	226.8
2	277.5	275.2	275.2	275.2
3	581.3	537.4	537.4	537.4
4	911.3	861.9	862.5	861.5
5	1049.4	918.8	968.3	974.8

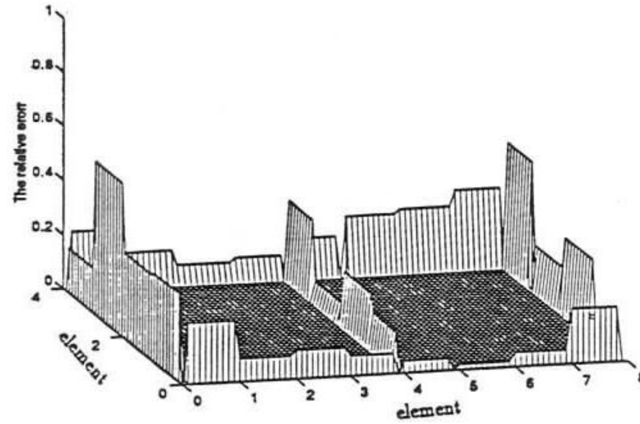

Fig. 4 The relative changes in the norm of the element stiffness matrices, Least-Squares solution (modes 1–3)

3 Experimental Case Study

In order to assess the performance of different structural model updating strategies, we provide a comparison between these strategies when applied to updating the finite element model of an actual structure.

The test structure used for the survey was required to be somewhat complicated without being unduly extensive. The design chosen was the in-plane welded frame structure illustrated in Fig. 1. The frame contains four "L" shape joints and two "T" shape joints which are difficult to model. The frame is made of *25.4 mm (1 inch)* square aluminum tubing with *2.38 mm ($\frac{3}{32}$ inch)* wall thickness.

The modal analysis test was performed under free-free boundary conditions; the structure was suspended from coordinate 7 using a wire. An accelerometer was mounted at coordinate 1, and the structure was excited using a hammer at 13 locations shown in Fig. 1. The first five strain modes of the structure were measured.

The finite element model of the frame consists of 28 in-plane frame elements. The inplane frame element is a combination of a beam element and a rod element. The beam part is modeled using Euler-Bernoulli beam theory. The displacement vector of the element is:

$$\left[w_{i-1}, L\frac{dw_{i-1}}{dx}, L\theta_{i-1}, w_i, L\frac{dw_i}{dx}, L\theta_i \right].$$

where w_i, dw_i/dx and θ_i are, respectively, the transverse dis-

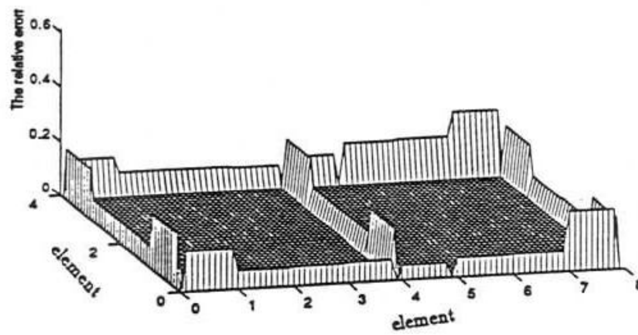

Fig. 5 The relative changes in the norm of the element stiffness matrices, Adjusted solution (modes 1–3)

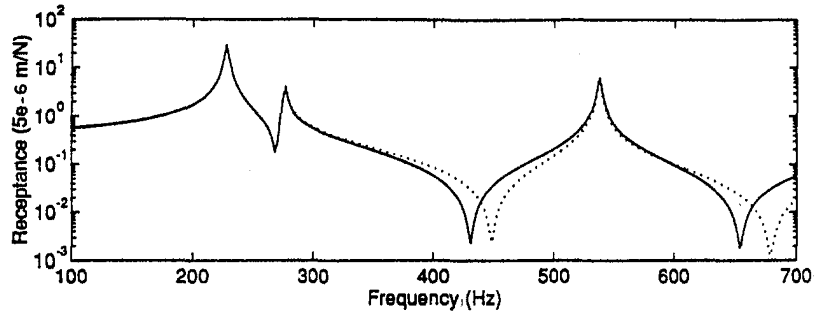

Fig. 6 FRF (6Z/6Z) of the test and Baruch's models

placement, the bending slope and the twisting angle at the i^{th} node. L is the length of the element.

The stiffness and consistent mass matrices used to develop the finite element model are:

$$\mathbf{K}_0 = k_0 \begin{bmatrix} 12 & 6 & 0 & -12 & 6 & 0 \\ & 4 & 0 & -6 & 2 & 0 \\ & & r & 0 & 0 & -r \\ & & & 12 & -6 & 0 \\ & Sym. & & & 4 & 0 \\ & & & & & r \end{bmatrix},$$

$$\mathbf{M}_0 = \frac{m_0}{420} \begin{bmatrix} 156 & 22 & 0 & 54 & -13 & 0 \\ & 4 & 0 & 13 & -3 & 0 \\ & & r_g & 0 & 0 & r_g/2 \\ & & & 156 & -22 & 0 \\ & Sym. & & & 4 & 0 \\ & & & & & r_g \end{bmatrix},$$

where m_0 and k_0 are the mass (ρAL) and the flexural rigidity (EI/L^3) of the element, $r = GJ_t/EI$ is the ratio between torsional rigidity and flexural rigidity for unit length, and $r_g = J/AL^2$ is the polar moment of inertia of the frame cross section non-dimensionalized using the area of the cross section and the length of the element.

Table 1 and Fig. 2 show that there is a considerable discrepancy between the FE and test results. The measured mode shapes exhibited little scatter in the phase angle; they were converted to real modes by taking the modulus of each mode shape coefficient and multiplying it by the sign of its real part. These real modes were used to update the FE model.

One of the major difficulties in model updating is the incompatibility between the number of degrees of freedom of the finite element and test models. It is impractical to acquire data from all the degrees of freedom of the structure which are used in the finite element model. There are two ways of overcoming this difficulty:

- reducing the size of the finite element model to fit the number of DOF used in the test, or
- using the finite element model to obtain values for the components of the mode shapes which are not known from testing. This is called *expanding* the mode shapes.

As a first step in the updating we examined the FE mass matrix. We expanded the mode shapes, supposing that we knew a part, $\phi_i^{(1)}$ of the i^{th} mode, and finding the remainder, $\phi_i^{(2)}$; by requiring

$$\min_{\phi_i^{(2)}} \left\| (K_0 - \lambda_i \mathbf{M}_0) \begin{Bmatrix} \boldsymbol{\phi}_i^{(1)} \\ \boldsymbol{\phi}_i^{(2)} \end{Bmatrix} \right\| \quad i = 1, \ldots, 5. \tag{15}$$

where $\mathbf{M}_0$ and $\mathbf{K}_0$ are the mass and stiffness matrices of the FE model. We mass normalized the extended modes so formed and computed the generalized mass matrix, $\Phi^T\mathbf{M}_0\Phi$, for the first five expanded modes. We found

$$\Phi^T\mathbf{M}_0\Phi = \begin{bmatrix} 1 & -0.0221 & 0.0207 & 0.0133 & 0.0105 \\ & 1 & 0.0036 & -0.0001 & -0.0096 \\ & & 1 & -0.0171 & -0.0022 \\ & Sym. & & 1 & 0.0064 \\ & & & & 1 \end{bmatrix},$$

We interpreted the near-orthogonality of the modes with respect to $\mathbf{M}_0$ as evidence that our measurements were accurate, and that $\mathbf{M}_0$ adequately represented the mass matrix of the structure.

Having ascertained that $\mathbf{M}_0$ was adequate, we revised the way in which we extended the measured modes; we extended them so that the extended modes would be precisely orthogonal with respect to $\mathbf{M}_0$, as follows.

Denote the truncated measured modes by $\tilde{\Phi} = [\phi_1^{(1)}, \ldots, \phi_m^{(1)}]$, and the corresponding truncated modes from the FE model by $\tilde{\Phi}_0$. Normalize columns of $\tilde{\Phi}$ so that they have the same length as their analytical counterparts.

Now find the orthogonal matrix $\mathbf{R}$ which approximately rotates $\tilde{\Phi}$ to $\tilde{\Phi}_0$ by taking

$$\min_{\mathbf{R}} \|\tilde{\Phi}\mathbf{R} - \tilde{\Phi}_0\|, \quad Subject\ to \quad \mathbf{R}^T\mathbf{R} = \mathbf{I}. \tag{16}$$

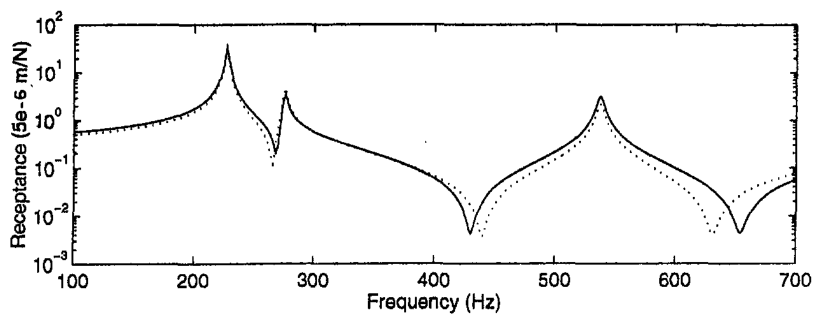

Fig. 7 FRF (6Z/6Z) of the test and Kabe's updated model

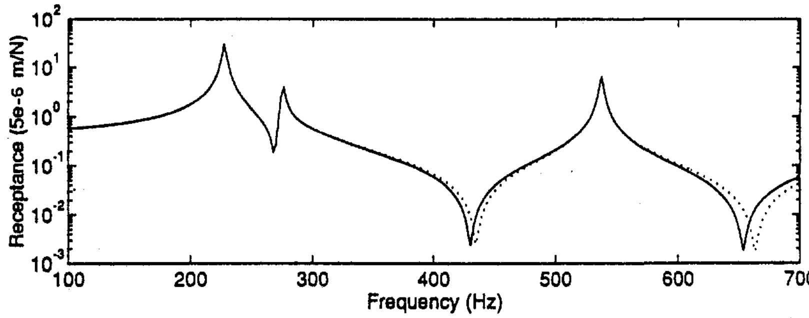

Fig. 8 FRF (6Z/6Z) of the test and our updated models

R may be found by SVD: if

$$\tilde{\boldsymbol{\Phi}}^T\tilde{\boldsymbol{\Phi}}_0 = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$$

then $\mathbf{R} = \mathbf{V}\mathbf{U}^T$ (Golub and Van Loan, 1991).

Now we expand the measured modes $\tilde{\boldsymbol{\Phi}}$ to full modes $\boldsymbol{\Phi}$ by rotating the full FE modes $\boldsymbol{\Phi}_0$ with the matrix **R**:

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}_0\mathbf{R}^T \tag{17}$$

These new extended modes will be orthogonal with respect to $\mathbf{M}_0$ because

$$\boldsymbol{\Phi}^T\mathbf{M}_0\boldsymbol{\Phi} = \mathbf{R}\boldsymbol{\Phi}_0^T\mathbf{M}_0\boldsymbol{\Phi}_0\mathbf{R}^T = \mathbf{R}\mathbf{I}\mathbf{R}^T = \mathbf{R}\mathbf{R}^T = \mathbf{I}.$$

It is these expanded modes and the measured frequencies which are used for the model updating.

4 Performance of Updating Procedures

In this section we update the stiffness matrix of the frame structure using the various methods: matrix updating; matrix updating maintaining the pattern of zeros in the model; physical parameter updating; using generic stiffness matrices. In all of these procedures the model parameters are adjusted by forming an equation error function using the first three quasi-measured modes; a set of linear equations is formed by rearranging the equation of motion and orthogonality requirements of the modes in terms of the updating parameters. When the number of parameters is more than the number of equations, we adjust the model by making the minimum changes in the parameters, otherwise a Least-Squares solution is performed. We judge the performance of each method by its ability to reproduce the first three measured modes; to predict the fourth and fifth measured modes; and more importantly, by its ability to predict the modes of the structure when there is a design change. The latter criterion is to ensure that the model corresponds to a physical system.

We identify the model of the test structure by an iterative procedure in which each iteration has two sub-steps:

(a) use the current estimate of **K**, along with $\mathbf{M}_0$ to obtain the ϕ_i using the analysis in Section 3.

(b) use the obtained ϕ_i to compute a new estimate of **K**, using the analysis described before.

In the first application of sub-step (a) we use the finite element model $\mathbf{K}_0$. We found that all the procedures converged (in the sense that there was no further change) after 2–4 iterations; we report results obtained after 5 iterations.

4.1 Matrix Updating Method. Baruch and Bar Itzhack (1978) updated $\mathbf{K}_0$ by making the minimum symmetric changes in entries of the stiffness matrix so that the model was consistent with the test results, i.e.,

$$\mathit{min}\ \|\mathbf{M}_0^{-1/2}(\mathbf{K} - \mathbf{K}_0)\mathbf{M}_0^{-1/2}\|,$$

$$\mathit{subject\ to}\quad \mathbf{K}\boldsymbol{\Phi} = \mathbf{M}_0\boldsymbol{\Phi}\boldsymbol{\Lambda},\quad \boldsymbol{\Phi}^T\mathbf{K}\boldsymbol{\Phi} = \boldsymbol{\Lambda},\quad \mathit{and}\quad \mathbf{K} = \mathbf{K}^T.$$

The final equation in the procedure is a closed form solution for the updated stiffness matrix:

$$\mathbf{K} = \mathbf{K}_0 + \boldsymbol{\Delta} + \boldsymbol{\Delta}^T, \tag{18}$$

where

$$\boldsymbol{\Delta} = (\mathbf{I} - \mathbf{M}_0\boldsymbol{\Phi}\boldsymbol{\Phi}^T/2)(\mathbf{M}_0\boldsymbol{\Phi}\boldsymbol{\Lambda} - \mathbf{K}_0\boldsymbol{\Phi})\boldsymbol{\Phi}^T\mathbf{M}_0.$$

Table 2 shows the natural frequencies of the updated model after 5 iterations. The first three measured modes were used in updating; the fourth and fifth were predicted. Figure 3 shows the difference between the final updated stiffness matrix constructed via Baruch's method, and the original finite element stiffness matrix. The differences are spread all over the stiffness matrix, and it is not possible to identify meaningful element stiffness matrices in the updated global stiffness matrix.

We notice that except for the modes used in updating, Baruch's model has the same eigendata as the original finite element model. To see why in this example the updating procedure corresponds to a mixed eigenvalue problem, let us rewrite the objective function of the Baruch's method as

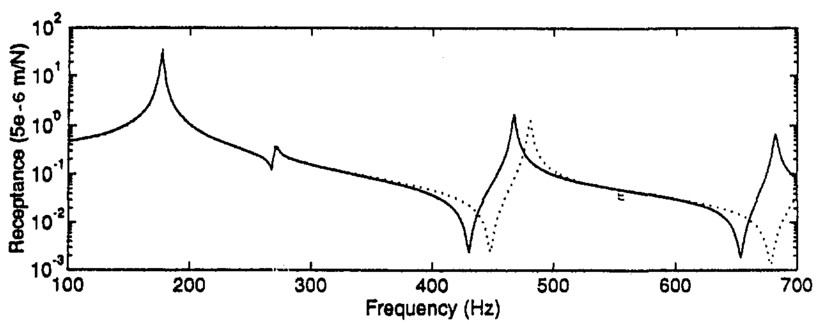

Fig. 9 FRF (6Z/6Z) of the modified structure and Baruch's modified model

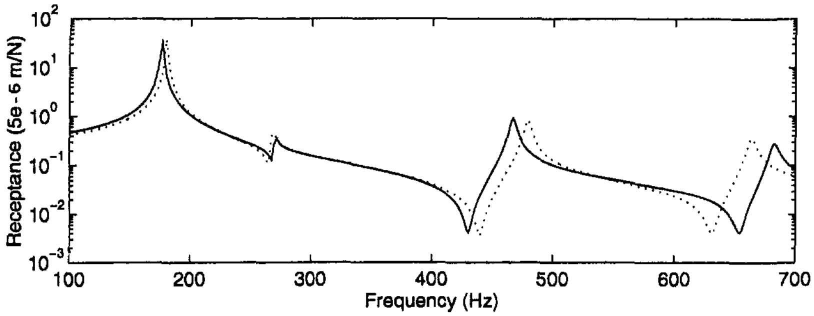

Fig. 10 FRF (6Z/6Z) of the modified structure and the Kabe's modified model

$$
\begin{aligned}
\|\mathbf{M}_0^{-1/2}(\mathbf{K}-\mathbf{K}_0)\mathbf{M}_0^{-1/2}\| &= \|\mathbf{M}_0^{-1/2}\Big(\sum_{i=1}^{N}\lambda_i\boldsymbol{\phi}_i\boldsymbol{\phi}_i^T-\lambda_{0i}\boldsymbol{\phi}_{0i}\boldsymbol{\phi}_{0i}^T\Big)\mathbf{M}_0^{-1/2}\| \\
&= \|\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T-\mathbf{V}_0\boldsymbol{\Lambda}_0\mathbf{V}_0^T\|
\end{aligned}
$$

where

$$\mathbf{v}_i=\mathbf{M}_0^{1/2}\boldsymbol{\phi}_i,\quad \boldsymbol{\Lambda}=(\lambda_i)_{i=1}^{N},$$

$$\mathbf{v}_{0i}=\mathbf{M}_0^{1/2}\boldsymbol{\phi}_{0i},\quad \boldsymbol{\Lambda}_0=(\lambda_{0i})_{i=1}^{N},$$

and N is order of the model. The extended measured mode shapes have the same range as their analytical counterparts, it follows that

$$\mathbf{V}=\mathbf{V}_0\begin{bmatrix}\mathbf{R}_m & 0\\ 0 & \mathbf{Q}_{N-m}\end{bmatrix},$$

where $\mathbf{R}_m$ and $\mathbf{Q}_{N-m}$ are orthogonal rotation matrices. Now the objective function can be rewritten as

$$min\left\|\mathbf{V}_0\begin{bmatrix}\mathbf{R}_m\boldsymbol{\Lambda}_m\mathbf{R}_m^T-\boldsymbol{\Lambda}_{0_m} & \mathbf{0}\\ \mathbf{0} & \mathbf{Q}_{N-m}\boldsymbol{\Lambda}_{N-m}\mathbf{Q}_{N-m}^T-\boldsymbol{\Lambda}_{0_{N-m}}\end{bmatrix}\mathbf{V}_0^T\right\|.$$

Matrices $\boldsymbol{\Lambda}_m$ and $\mathbf{R}_m$ are known, thus the minimum of the above function is achieved by setting $\mathbf{Q}_{N-m}=\mathbf{I}_{N-m}$ and $\boldsymbol{\Lambda}_{N-m}=\boldsymbol{\Lambda}_{0_{N-m}}$. This simply means that the updated model is consistent with the test results, and beyond that its eigendata is the same as that of the finite element model.

We now update the stiffness matrix using the other matrix updating method. This method keeps the pattern of zeros in the updated stiffness matrix the same as finite element model. This was suggested by Kabe (1985) and others, see for example Caesar and Peter (1987), Kammer (1988), and O'Callahan and Wu (1991). There are 231 non-zero entries in the upper triangular of $\mathbf{K}_0$ while the number of equations formed using the equations of motion and orthogonality of the modes, for the three chosen modes, is 249; the procedure is over-determined. Table 2 shows the first 8 modes (3 rigid body modes, 5 others) of the updated model. The model reproduces the input data exactly, but produces results for the last two modes which show no correlation with the measurements, and in fact are worse than those obtained from original FE model. The stiffness matrix is indefinite and there are non-zeros and even imaginary frequencies instead of the required zeros corresponding to rigid body modes.

The global stiffness matrix may be made positive semi-definite by introducing the three rigid body modes as data, in addition to the three measured modes. The results corresponding to this are shown in column 5 of Table 2. Still the predicted fifth frequency has no relation to the experimental value. A close inspection of the updated stiffness matrix shows that the model could be obtained by adjusting the parameters of a generic beam stiffness matrix and a generic rod stiffness matrix, with no coupling, at the element level.

4.2 Physical Parameter Updating. In physical parameter updating methods it is assumed that the modeling is correct and the problem is just to find the correct numerical values that should be used in the finite element model. This corresponds to accepting the matrices arising from (numerical) integration of finite element shape functions and modifying only the physical parameters EI/L^3, GJ_t/L in each element. Table 3 shows the results of updating using such a strategy. The updated model has the correct definiteness properties, but is little better than the original FE model in predicting the measured frequencies. The basic defect in the procedure is its inability to model the joints in the structure.

The next method which we used to identify the stiffness matrix of the frame structure was the one proposed by Nobari, Robb and Ewins (1993); for convenience we shall call it Nobari's method. To increase the flexibility of the updating procedure, especially for the joints, they considered a separate modification factor for each consistent degree of freedom involved in the element mass and stiffness matrices.

As an example, for the stiffness matrix of a frame element, with an initial model

$$\mathbf{K}_0^e=\frac{EI}{L^3}\begin{bmatrix}12 & 6L & 0 & -12 & 6L & 0\\ & 4L^2 & 0 & -6L & 2L^2 & 0\\ & & r & 0 & 0 & -r\\ & & & 12 & -6L & 0\\ & Sym. & & & 4L^2 & 0\\ & & & & & r\end{bmatrix}, \tag{19}$$

they selected four parameters k_1, k_2, k_3, k_4 and wrote,

$$
\begin{aligned}
\mathbf{K}^e = k_1&\begin{bmatrix}1 & 0 & 0 & -1 & 0 & 0\\ & 0 & 0 & 0 & 0 & 0\\ & & 0 & 0 & 0 & 0\\ & & & 1 & 0 & 0\\ & Sym. & & & 0 & 0\\ & & & & & 0\end{bmatrix}\\
+ k_2&\begin{bmatrix}0 & 1 & 0 & 0 & 1 & 0\\ & 0 & 0 & -1 & 0 & 0\\ & & 0 & 0 & 0 & 0\\ & & & 0 & -1 & 0\\ & Sym. & & & 0 & 0\\ & & & & & 0\end{bmatrix}\\
+ k_3&\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ & 2 & 0 & 0 & 1 & 0\\ & & 0 & 0 & 0 & 0\\ & & & 0 & 0 & 0\\ & Sym. & & & 1 & 0\\ & & & & & 0\end{bmatrix}
\end{aligned}
$$

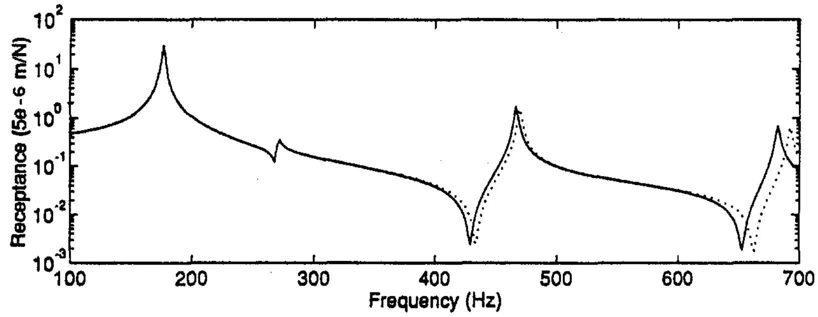

Fig. 11 FRF (6Z/6Z) of the modified structure and our modified model

$$+ k_4 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & -1 \\ & & & 0 & 0 & 0 \\ & Sym. & & & 0 & 0 \\ & & & & & 1 \end{bmatrix} . \quad (20)$$

For the in-plane frame element (19) the parameters have the values

$$k_1 = \frac{12EI}{L^3}, \quad k_2 = \frac{6EI}{L^2}, \quad k_3 = \frac{2EI}{L}, \quad k_4 = \frac{EIr}{L^3} = \frac{GJ_t}{L}.$$

For the present finite element model with 28 elements there are thus $4 \times 28 = 112$ unknowns to be identified. Table 3 illustrates the natural frequencies of the updated model via this method.

The results show improvements in predicting the higher modes. But the updated model does not represent any physical system; it has an imaginary frequency and only one zero frequency, instead of the required three zero frequencies. This discrepancy arises at the element level; the beam part of the element does not have the correct two rigid body modes. For it to do so we must have

$$\begin{bmatrix} k_1 & k_2 & -k_1 & k_2 \\ & 2k_3 & -k_2 & k_3 \\ & Sym. & k_1 & -k_2 \\ & & & 2k_3 \end{bmatrix} \begin{bmatrix} 1 & -L/2 \\ 0 & 1 \\ 1 & L/2 \\ 0 & 1 \end{bmatrix} = 0.$$

This holds only if

$$k_2 = k_1 L/2, \quad k_3 = k_1 L^2/6. \quad (21)$$

This means that if we write $k_1 = 12EI/L^3$ then we have

$$k_2 = \frac{6EI}{L^2}, \quad k_3 = \frac{2EI}{L}.$$

Thus the Nobari's model is a physically meaningful model **only if it is the ordinary FEM model** (19).

To verify this statement we entered the three overall rigid body modes as data in Nobari's model and found that it predicted the same results as those obtained by physical parameter updating, shown in the third column of Table 3.

4.3 Generic Stiffness Matrices. We updated the stiffness matrix of the frame by modifying the eigendata of its element stiffness matrices as discussed in Section 2. Each element stiffness matrix has order six and rank three. In general, it may be defined using six parameters:

$$\mathbf{K}^e = \mathbf{U}_0 \mathbf{R} \Lambda \mathbf{R}^T \mathbf{U}_0^T = \mathbf{U}_0 \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ & k_{22} & k_{23} \\ & & k_{33} \end{bmatrix} \mathbf{U}_0^T,$$

where

$$\mathbf{U}_0^T = \begin{bmatrix} 0 & \alpha & 0 & 0 & -\alpha & 0 \\ 2\beta & \beta & 0 & -2\beta & \beta & 0 \\ 0 & 0 & \alpha & 0 & 0 & -\alpha \end{bmatrix}, \quad \begin{matrix} \alpha = \sqrt{2}/2 \\ \beta = \sqrt{10}/10 \end{matrix}$$

The diagonal terms k_{11}, k_{22} and k_{33} represent, respectively, the effects of bending, shear and twisting modes in the element, while the off diagonal terms, k_{12}, k_{13}, k_{23}, account for the coupling effects between these modes. The first strain mode of the element is symmetric, while the second and third modes are antisymmetric. Thus for any symmetrical frame element, i.e. not a joint element, k_{12} and k_{23} must be zero. Of the 28 elements, 14 are joints, with 6 parameters k_{ij} each; 14 are symmetric, with 4 parameters each; there are thus 140 parameters to be updated.

We introduced only the first three quasi-measured modes into the identification procedure, and obtained a Least-Squares solution; the natural frequencies of the resultant model are shown in Table 4. The LS solution reproduces the correct rigid body modes. Also the other modes are much closer to the test results than those of previous models. This shows the importance of proper selection of the updating parameters.

To understand the results of the Least-Squares solution, we put the relative norm of the change (from the FE model to the final model) in stiffness of each element, $\|\Delta \mathbf{K}^e\|/\|\mathbf{K}_0^e\|$, on top of the element in the frame structure as shown in Fig. 4. The Least-Squares solution indicates that the errors in the original model are located in the connecting beams as well as in the joints.

There is an infinite number of stiffness matrices that reproduce the specified modal data, and our unbiased LS solution is just one of them. To increase the confidence in the updating procedure outcome, we may use the available *a priori* knowledge about the structure in the updating and try to filter out the noise effects. One way of achieving this is to adjust the solution by requiring that similar elements have similar models. The elements can be grouped into three sets: *T* joint elements, *L* joint elements, and connecting elements. These requirements were inserted into the updating procedure and an adjusted solution was obtained; the results are listed in Table 4. As expected, there is no change within the first three modes of the model after adjustment, but by looking at where mis-modeling in the original model occurred in Fig. 5, we notice that the adjusted solution and the Least-Squares solutions are different; the adjusted solution locates the errors mainly in the joints.

To assess the updated models further, we examined the receptance (FRF at 6Z/6Z) synthesized from the updated models and compared it with the test data. The measured damping ratios are used in the computations of the receptances. Figures 6–8 shows the computed FRF's (dotted lines), superimposed on the measured data (solid lines).

Figure 6 shows that Baruch's model follows the three resonances very well, while the antiresonances are shifted to the right. The Kabe's updated model, Fig. 7, also follows the three resonance frequencies exactly, but the second antiresonance is shifted. Figure 8 shows that our updated model is the closest to the test data throughout the displayed frequency range.

We introduced a design change into the test model, and evaluated these updated models in terms of the inaccuracy in predicting the effects of the design change. The design change involved adding a lumped mass at coordinate 6 and grounding the structure from this coordinate using a spring. This modification shifts the fourth mode of the structure below *700 Hz*. Figures 9–11 show the predictions of different models superimposed on the modified structure response. Both models resulted from matrix updating methods yield an inaccurate third mode. The fourth mode is in error in the modified Kabe's model and it does not appear in Baruch's model within the range of interest. Our model on the other hand gives much better prediction of the fourth mode, a slightly inaccurate third mode, and the exact first two modes.

In summary, our model produces the original as well as the modified structure response more closely than the other methods investigated in this paper.

5 Conclusion

An experimental survey was performed to assess the abilities of different updating procedures. It was shown that the success of updating procedures in reconstructing a physical model consistent with the modal testing data depended on the way the model parameters were selected. Updating the model by adjusting all the (non-zero) entries of the finite element model in an optimization procedure (matrix updating), yields a model consistent with the test data, but the model may not correspond to a physical structure. On the other hand, adjusting only the physical parameters does not produce a model consistent with the test data. The answer appears to lie in defining a generic model for each element and minimizing the error function by adjusting the acceptable model parameters.

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