Steady state heat flow through long hollow circular cylinders can be described by the following ordinary differential equation.

$$\frac{d}{dr}\left(kA\frac{dT(r)}{dr}\right) + AQ = 0 \qquad r_i < r < r_0$$

$$T(r_i) = T_i\,; \quad T(r_0) = T_0$$

where r is the radial coordinate, $T(r)$ is the temperature, k is the thermal conductivity, Q is the heat generation per unit area, $A = 2\,\pi\,r\,L$ is the surface area, L is the length of the cylinder, r_i is the inner radius, and r_0 is the outer radius. The boundary conditions specify the temperature on the inside and outside of the cylinder respectively.

(a) Show that the following represents an exact solution for the problem for the case when $Q = 0$.

$$T(r) = T_i - (T_i - T_0)\frac{\ln(r/r_i)}{\ln(r_0/r_i)}$$

(b) Show that the following is an appropriate weak form for obtaining an approximate solution using the Galerkin weighted residual method.

$$\int_{r_i}^{r_0}\left(-kA\frac{dT}{dr}\frac{dw_i}{dr} + AQw_i\right)dr = 0$$

Where w_j, $j = 1, 2, \ldots$ are the Galerkin weighting functions.

(c) Using the weak form given in (b) find an approximate solution of the problem with a trial solution of the form $T(r) = a_0 + a_1\,r + a_2\,r^2$. Assume the following numerical values: r_i = 1 in, r_0 = 4 in, T_i = 400 °F, T_i = 100 °F, L = 100 in, Q = 0, k = 0.04 Btu/(h.ft.°F).

Prob #4

(a) $$\frac{d}{dr}\left(KA\frac{dT}{dr}\right) = 0$$

$$\Rightarrow \quad KA\frac{dT}{dr} = C_1 \qquad (A = 2\pi r L$$

$$\frac{dT}{dr} = \frac{C_1}{2K\pi L r}$$

$$\Rightarrow \quad T(r) = \frac{C_1}{2\pi K L}\ln r + C_2$$

BCs: $\begin{cases} T(r_i) = T_i \\ T(r_o) = T_o \end{cases} \Rightarrow \begin{cases} \frac{1}{2\pi KL}\ln r_i \; C_1 + C_2 = T_i \\ \frac{1}{2\pi KL}\ln r_o \; C_1 + C_2 = T_o \end{cases}$

get: $\begin{cases} C_1 = 2\pi KL(T_o - T_i) / \ln(r_o/r_i) \\ C_2 = (T_i \ln r_o - T_o \ln r_i) / \ln(r_o/r_i) \end{cases}$

$$\Rightarrow \quad T(r) = \frac{T_o - T_i}{\ln r_o/r_i}\ln r + (T_i \ln r_o - T_o \ln r_i)/\ln(r_o/r_i)$$

$$= \frac{(T_o - T_i)\ln r - (T_o - T_i)\ln r_i + T_i(\ln r_i - \ln r_o)}{\ln(r_o/r_i)}$$

$$\Rightarrow \quad \boxed{T(r) = T_i - (T_i - T_o)\frac{\ln(r/r_i)}{\ln(r_o/r_i)}}$$

(b) Weighted residual:

$$\int_{r_i}^{r_o} W_i \left[\frac{d}{dr}\left(KA\frac{dT}{dr}\right) + AQ\right] dr = 0$$

$\underline{IBP}$:

$$\left[KA W_i \frac{dT}{dr}\right]_{r_i}^{r_o} - \int_{r_i}^{r_o} KA \frac{dT}{dr}\frac{dW_i}{dr}\, dr + \int_{r_i}^{r_o} W_i AQ\, dr = 0$$

But: $W_i(r_o) = W_i(r_i) = 0$ (EBCs)

get:

$$\boxed{\int_{r_i}^{r_o}\left(-KA\frac{dT}{dr}\frac{dW_i}{dr} + AQW_i\right) dr = 0}$$ Weak form

(C) $T(r)$ satisfies the EBCs:

$$\begin{cases} T(r_i) = T_i = a_0 + a_1 r_i + a_2 r_i^2 \\ T(r_o) = T_o = a_0 + a_1 r_o + a_2 r_o^2 \end{cases}$$

get:

$$\begin{cases} a_0 = \dfrac{r_o T_i - r_i T_o}{r_o - r_i} + a_2 r_i r_o \\ a_1 = \dfrac{T_o - T_i}{r_o - r_i} - (r_i + r_o) a_2 \end{cases}$$

get:

$$\boxed{T(r) = \left[\frac{r_o T_i - r_i T_o}{r_o - r_i} + a_2 r_o r_i\right] + \left[\frac{T_o - T_i}{r_o - r_i} - (r_o + r_i) a_2\right] r + a_2 r^2}$$

take: $r_i = 1$ $r_o = 4$

$T_i = 400$ $T_o = 100$

$$T(r) = \left(\frac{1600 - 100}{4-1} + 4a_2\right) + \left(\frac{-300}{3} - 5a_2\right) r + a_2 r^2$$

$$\boxed{T(r) = 500 + 4a_2 - (100 + 5a_2) r + a_2 r^2}$$

Galerkin : $W(r) = \frac{dT(r)}{da_2} = -5r + r^2$

$$\frac{dW}{dr} = -5 + 2r$$

& $\frac{dT}{dr} = -(100 + 5a_2) + 2a_2 r$

Note $\underset{K=}{Q=0.}$ the weak form becomes:

$A = 2\pi r L$

$$\int_{r_i}^{r_o} -KA(-100-5a_2+2a_2 r)(-5+2r)\,dr = 0$$

or: $$\int_{r_i}^{r_o} r(-100-5a_2+2a_2 r)(-5+2r)\,dr = 0$$

$\Rightarrow$ $$\int_{r_i=1}^{r_o=4} \left[-200r^2 - 10r^2 a_2 + 4a_2 r^3 + 500r + 25a_2 r - 10r^2 a_2\right] dr = 0$$

$\Rightarrow$ $$-200 \cdot \frac{1}{3}(4^3-1^3) - 10a_2 \cdot \frac{1}{3}(4^3-1^3) + 4a_2 \cdot \frac{1}{4}[4^4-1^4] + 500 \times \frac{1}{2}[16-1]$$
$$+25a_2 \cdot \frac{1}{2}[4^2-1] - 10a_2 \cdot \frac{1}{3}[4^3-1] = 0$$

$$-4200 - 210a_2 + 255a_2 + 3750 + 187.5a_2 - 210a_2 = 0$$

$$-450 + 22.5a_2 = 0 \quad \Rightarrow \quad \boxed{a_2 = 20}$$

get:

$$\boxed{T(r) = 20r^2 - 200r + 580}$$