

Two-Dimensional Problems Using Constant Strain Triangles

5.1 INTRODUCTION

The two-dimensional finite element formulation in this chapter follows the steps used in the one-dimensional problem. The displacements, traction components, and distributed body force values are functions of the position indicated by (x, y) . The displacement vector \mathbf{u} is given as

$$\mathbf{u} = [u, v]^T \quad (5.1)$$

where u and v are the x and y components of \mathbf{u} , respectively. The stresses and strains are given by

$$\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \tau_{xy}]^T \quad (5.2)$$

$$\boldsymbol{\epsilon} = [\epsilon_x, \epsilon_y, \gamma_{xy}]^T \quad (5.3)$$

From Fig. 5.1, representing the two-dimensional problem in a general setting, the body force, traction vector, and elemental volume are given by

$$\mathbf{f} = [f_x, f_y]^T \quad \mathbf{T} = [T_x, T_y]^T \quad \text{and} \quad dV = t dA \quad (5.4)$$

where t is the thickness along the z direction. The body force \mathbf{f} has the units force/unit volume, while the traction force \mathbf{T} has the units force/unit area. The strain-displacement relations are given by

$$\boldsymbol{\epsilon} = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right]^T \quad (5.5)$$

Stresses and strains are related by (see Eqs. 1.18 and 1.19)

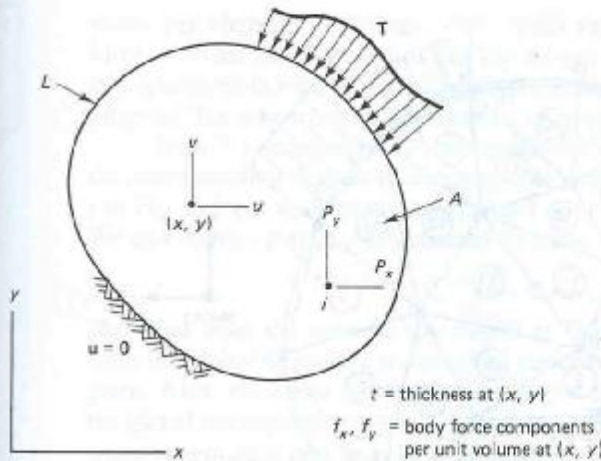


Figure 5.1 Two-dimensional problem.

$$\sigma = D\epsilon \quad (5.6)$$

The region is discretized with the idea of expressing the displacements in terms of values at discrete points. Triangular elements are introduced first. Stiffness and load concepts are then developed using energy and Galerkin approaches.

5.2 FINITE ELEMENT MODELING

The two-dimensional region is divided into straight-sided triangles. Figure 5.2 shows a typical triangulation. The points where the corners of the triangles meet are called *nodes*, and each triangle formed by three nodes and three sides is called an *element*. The elements fill the entire region except a small region at the boundary. This unfilled region exists for curved boundaries and it can be reduced by choosing smaller elements or elements with curved boundaries. The idea of the finite element method is to solve the continuous problem approximately, and this unfilled region contributes to some part of this approximation. For the triangulation shown in Fig. 5.2, the node numbers are indicated at the corners and element numbers are circled.

In the two-dimensional problem discussed here, each node is permitted to displace in the two directions x and y . Thus, each node has two degrees of freedom (dof's). As seen from the numbering scheme used in trusses, the displacement components of node j are taken as Q_{2j-1} in the x direction and Q_{2j} in the y direction. We denote the global displacement vector as

$$\mathbf{Q} = [Q_1, Q_2, \dots, Q_N]^T \quad (5.7)$$

where N is the number of degrees of freedom.

Computationally, the information on the triangulation is to be represented in the form of *nodal coordinates* and *connectivity*. The nodal coordinates are stored in a two-dimensional array represented by the total number of nodes and the two coor-

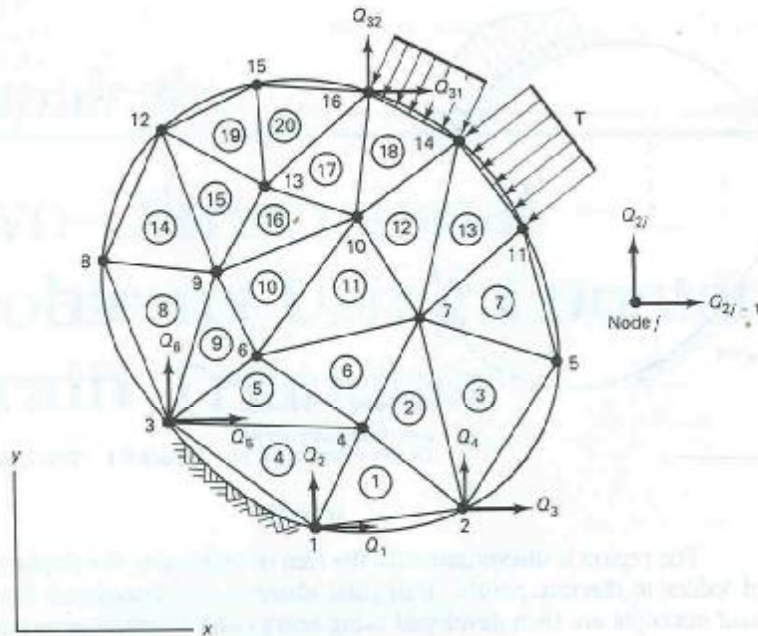


Figure 5.2 Finite element discretization.

ordinates per node. The connectivity may be clearly seen by isolating a typical element, as shown in Fig. 5.3. For the three nodes designated locally as 1, 2, and 3, the corresponding global node numbers are defined in Fig. 5.2. This element connectivity information becomes an array of the size of number of elements and three

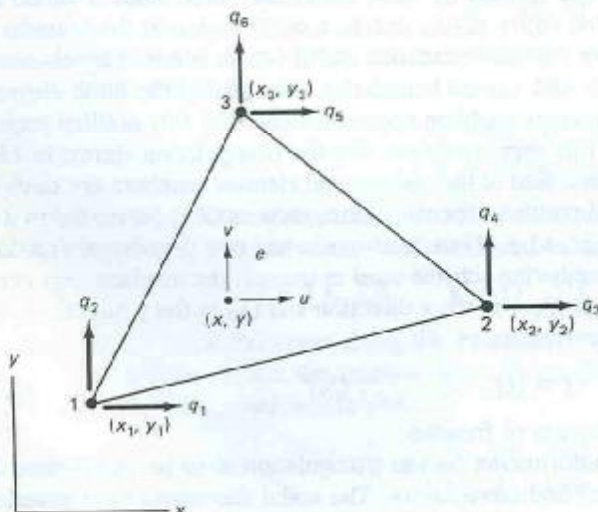


Figure 5.3 Triangular element.

nodes per element. A typical connectivity representation is shown in Table 5.1. Most standard finite element codes use the convention of going around the element in a counterclockwise direction to avoid calculating a negative area. However, in the program that accompanies this chapter, ordering is not necessary.

Table 5.1 establishes the correspondence of local and global node numbers and the corresponding degrees of freedom. The displacement components of a local node j in Fig. 5.3 are represented as q_{2j-1} and q_{2j} in the x and y directions, respectively. We denote the element displacement vector as

$$\mathbf{q} = [q_1, q_2, \dots, q_6] \quad (5.8)$$

Note that from the connectivity matrix in Table 5.1, we can extract the \mathbf{q} vector from the global \mathbf{Q} vector, an operation performed frequently in a finite element program. Also, the nodal coordinates designated by (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) have the global correspondence established through Table 5.1. The local representation of nodal coordinates and degrees of freedom provides a setting for a simple and clear representation of element characteristics.

TABLE 5.1 ELEMENT CONNECTIVITY

| Element number e | Three nodes | | |
|-----------------------|-------------|----|----|
| | 1 | 2 | 3 |
| 1 | 1 | 2 | 4 |
| 2 | 4 | 2 | 7 |
| ⋮ | | | |
| 11 | 6 | 7 | 10 |
| ⋮ | | | |
| 20 | 13 | 16 | 15 |

5.3 CONSTANT STRAIN TRIANGLE (CST)

The displacements at points inside an element need to be represented in terms of the nodal displacements of the element. As discussed earlier, the finite element method uses the concept of shape functions in systematically developing these interpolations. For the constant strain triangle, the shape functions are linear over the element. The three shape functions N_1 , N_2 , and N_3 corresponding to nodes 1, 2, and 3, respectively, are shown in Fig. 5.4. Shape function N_1 is 1 at node 1 and linearly reduces to 0 at nodes 2 and 3. The values of shape function N_1 thus define a plane surface shown shaded in Fig. 5.4a. N_2 and N_3 are represented by similar surfaces having values of 1 at nodes 2 and 3, respectively and dropping to 0 at the opposite edges. Any linear combination of these shape functions also represents a plane surface. In particular, $N_1 + N_2 + N_3$ represents a plane at a height of 1 at nodes 1, 2, and 3, and,

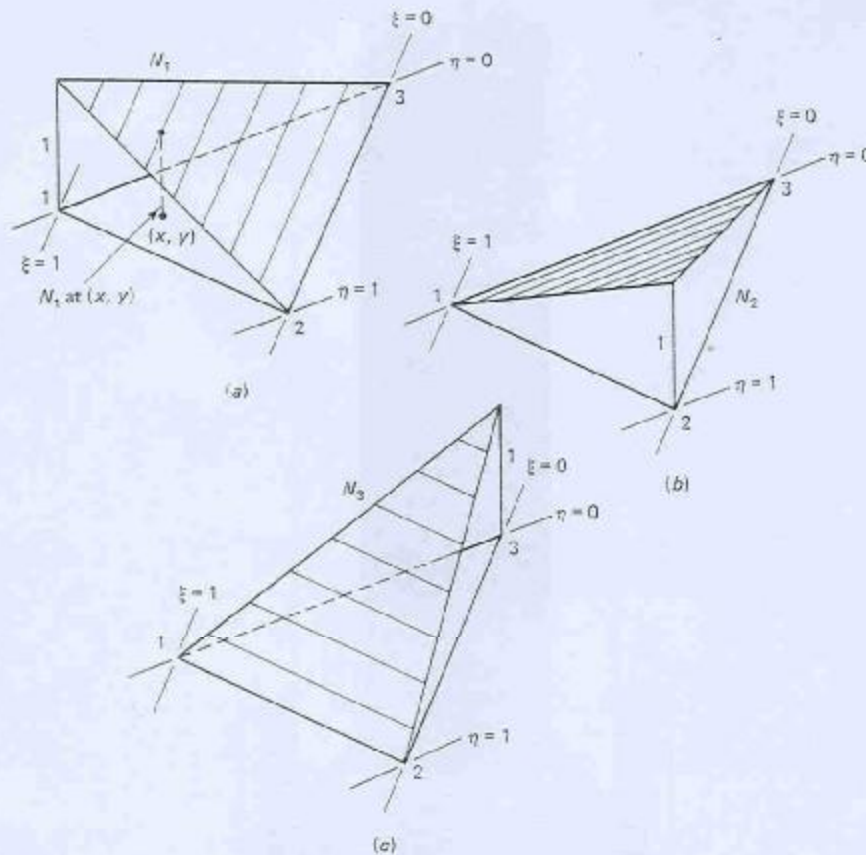


Figure 5.4 Shape functions.

thus, it is parallel to the triangle 123. Consequently, for every N_1 , N_2 , and N_3 ,

$$N_1 + N_2 + N_3 = 1 \quad (5.9)$$

N_1 , N_2 , and N_3 are therefore not linearly independent; only two of these are independent. The independent shape functions are conveniently represented by the pair ξ , η , as follows

$$N_1 = \xi \quad N_2 = \eta \quad N_3 = 1 - \xi - \eta \quad (5.10)$$

where ξ , η are natural coordinates (Fig. 5.4). At this stage, the similarity with the one-dimensional element (Chapter 3) should be noted: In the one-dimensional problem the x coordinates were mapped onto the ξ coordinates and shape functions were defined as functions of ξ . Here, in the two-dimensional problem, the x , y coordinates are mapped onto the ξ , η coordinates and shape functions are defined as functions of ξ and η .

The shape functions can be physically represented by **area coordinates**. A point (x, y) in a triangle divides it into three areas, A_1 , A_2 , and A_3 , as shown in Fig. 5.5. The shape functions N_1 , N_2 , and N_3 are precisely represented by

$$N_1 = \frac{A_1}{A} \quad N_2 = \frac{A_2}{A} \quad N_3 = \frac{A_3}{A} \quad (5.11)$$

where A is the area of the element. Clearly, $N_1 + N_2 + N_3 = 1$ at every point inside the triangle.

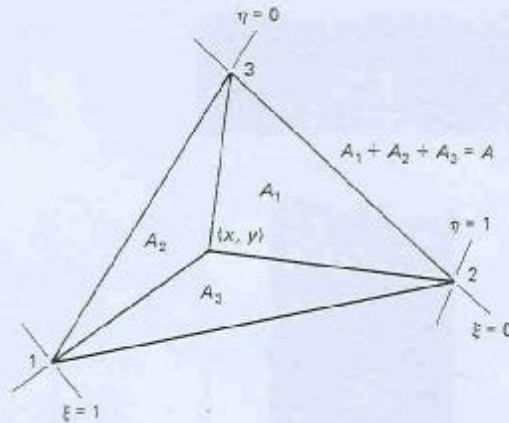


Figure 5.5 Area coordinates.

Isoparametric Representation

The displacements inside the element are now written using the shape functions and the nodal values of the unknown displacement field.

$$\begin{aligned} u &= N_1 q_1 + N_2 q_2 + N_3 q_3 \\ v &= N_1 q_4 + N_2 q_5 + N_3 q_6 \end{aligned} \quad (5.12a)$$

or using Eq. 5.10,

$$\begin{aligned} u &= (q_1 - q_3)\xi + (q_2 - q_3)\eta + q_3 \\ v &= (q_4 - q_6)\xi + (q_5 - q_6)\eta + q_6 \end{aligned} \quad (5.12b)$$

The relations 5.12a can be expressed in a matrix form by defining a shape function matrix \mathbf{N} ,

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \quad (5.13)$$

and

$$\mathbf{u} = \mathbf{N}\mathbf{q} \quad (5.14)$$

For the triangular element, the coordinates x, y can also be represented in terms of nodal coordinates using the same shape functions. This is **isoparametric representation**. This approach lends to simplicity of development and retains the uniformity with other complex elements. We have

$$x = N_1x_1 + N_2x_2 + N_3x_3 \quad (5.15a)$$

$$y = N_1y_1 + N_2y_2 + N_3y_3$$

or

$$x = (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3 \quad (5.15b)$$

$$y = (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3$$

Using the notation, $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$, we can write Eq. 5.15b as

$$x = x_{13}\xi + x_{23}\eta + x_3 \quad (5.15c)$$

$$y = y_{13}\xi + y_{23}\eta + y_3$$

This equation relates x and y coordinates to the ξ and η coordinates. Equation 5.12 expresses u and v as functions of ξ, η .

Example 5.1

Evaluate the shape functions N_1, N_2 , and N_3 at the interior point P for the triangular element shown in Fig. E5.1.

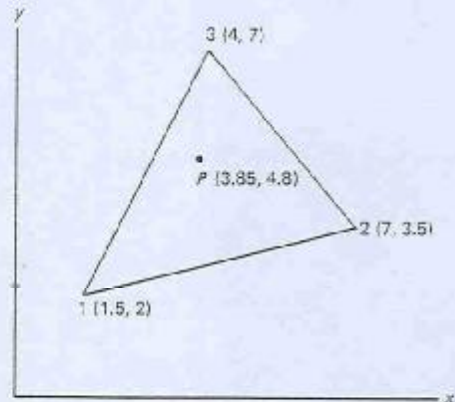


Figure E5.1 Examples 5.1 and 5.2.

Solution Using the isoparametric representation (Eqs. 5.15), we have

$$3.85 = 1.5N_1 + 7N_2 + 4N_3 = -2.5\xi + 3\eta + 4$$

$$4.8 = 2N_1 + 3.5N_2 + 7N_3 = -5\xi - 3.5\eta + 7$$

The two equations above are rearranged in the form

$$2.5\xi - 3\eta = 0.15$$

$$5\xi + 3.5\eta = 2.2$$

Solving the equations, we obtain $\xi = 0.3$ and $\eta = 0.2$, which implies

$$N_1 = 0.3 \quad N_2 = 0.2 \quad N_3 = 0.5$$

In evaluating the strains, partial derivatives of u and v are to be taken with respect to x and y . From Eqs. 5.12 and 5.15, we see that u , v and x , y are functions of ξ and η . That is, $u = u(x(\xi, \eta), y(\xi, \eta))$ and similarly $v = v(x(\xi, \eta), y(\xi, \eta))$. Using the chain rule for partial derivatives of u , we have

$$\begin{aligned} \frac{\partial u}{\partial \xi} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \xi} \\ \frac{\partial u}{\partial \eta} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \eta} \end{aligned}$$

which can be written in matrix notation as

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} \quad (5.16)$$

where the (2×2) square matrix is denoted as the Jacobian of the transformation, \mathbf{J} :

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad (5.17)$$

Some additional properties of the Jacobian are given in the Appendix. On taking the derivative of x and y ,

$$\mathbf{J} = \begin{bmatrix} x_{1\xi} & y_{1\xi} \\ x_{1\eta} & y_{1\eta} \end{bmatrix} \quad (5.18)$$

Also, from Eq. 5.16,

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} \quad (5.19)$$

where \mathbf{J}^{-1} is the inverse of the Jacobian \mathbf{J} , given by

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{2\xi} & -y_{1\xi} \\ -x_{2\xi} & x_{1\xi} \end{bmatrix} \quad (5.20)$$

$$\det \mathbf{J} = x_{1\xi}y_{2\xi} - x_{2\xi}y_{1\xi} \quad (5.21)$$

From the knowledge of the area of the triangle, it can be seen that the magnitude of

$\det \mathbf{J}$ is twice the area of the triangle. If the points 1, 2, and 3 are ordered in a counterclockwise manner, $\det \mathbf{J}$ is positive in sign. We have

$$A = \frac{1}{2} |\det \mathbf{J}| \quad (5.22)$$

where $|\cdot|$ represents the magnitude. Most computer codes use a counterclockwise order for the nodes and use $\det \mathbf{J}$ for evaluating the area.

Example 5.2

Determine the Jacobian of the transformation \mathbf{J} for the triangular element shown in Fig. E5.1.

Solution We have

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} -2.5 & -5.0 \\ 3.0 & -3.5 \end{bmatrix}$$

Thus, $\det \mathbf{J} = 23.75$ units. This is twice the area of the triangle. If 1, 2, 3 are in a clockwise order, then $\det \mathbf{J}$ will be negative. ■

From Eqs. 5.19 and 5.20, it follows that

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23} \frac{\partial u}{\partial \xi} - y_{13} \frac{\partial u}{\partial \eta} \\ -x_{23} \frac{\partial u}{\partial \xi} + x_{13} \frac{\partial u}{\partial \eta} \end{Bmatrix} \quad (5.23a)$$

Replacing u by the displacement v , we get a similar expression

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23} \frac{\partial v}{\partial \xi} - y_{13} \frac{\partial v}{\partial \eta} \\ -x_{23} \frac{\partial v}{\partial \xi} + x_{13} \frac{\partial v}{\partial \eta} \end{Bmatrix} \quad (5.23b)$$

Using the strain-displacement relations (5.5) and Eqs. 5.12b and 5.23, we get

$$\begin{aligned} \boldsymbol{\epsilon} &= \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \\ &= \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}(q_1 - q_5) - y_{13}(q_3 - q_5) \\ -x_{23}(q_2 - q_6) + x_{13}(q_4 - q_6) \\ -x_{23}(q_1 - q_5) + x_{13}(q_3 - q_5) + y_{23}(q_2 - q_6) - y_{13}(q_4 - q_6) \end{Bmatrix} \end{aligned} \quad (5.24a)$$

From the definition of x_{ij} and y_{ij} , we can write $y_{31} = -y_{13}$ and $y_{12} = y_{13} - y_{23}$, and so on. The above equation can be written in the form

$$\epsilon = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}q_1 + y_{31}q_3 + y_{12}q_5 \\ x_{32}q_2 + x_{13}q_4 + x_{21}q_6 \\ x_{32}q_1 + y_{23}q_2 + x_{13}q_3 + y_{31}q_4 + x_{21}q_5 + y_{12}q_6 \end{Bmatrix} \quad (5.24b)$$

The above equation can be written in matrix form as

$$\epsilon = \mathbf{B}q \quad (5.25)$$

where \mathbf{B} is a (3×6) element strain–displacement matrix relating the three strains to the six nodal displacements and is given by

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \quad (5.26)$$

It may be noted that all the elements of the \mathbf{B} matrix are constants expressed in terms of the nodal coordinates.

Example 5.3

Find the strain–nodal displacement matrices \mathbf{B}^e for the elements shown in Fig. E5.3. Use local numbers given at the corners.

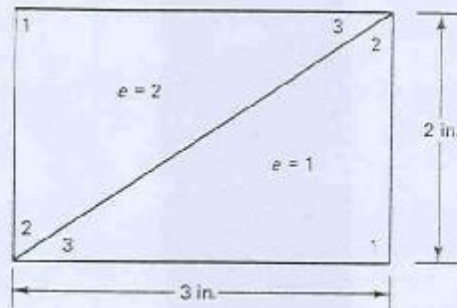


Figure E5.3

Solution We have

$$\begin{aligned} \mathbf{B}^e &= \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix} \end{aligned}$$

where $\det \mathbf{J}$ is obtained from $x_{13}y_{23} - x_{23}y_{13} = (3)(2) - (3)(0) = 6$. Using the local numbers at the corners, \mathbf{B}^e can be written using the relationship as

$$\mathbf{B}^2 = \frac{1}{6} \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix} \quad \blacksquare$$

Potential Energy Approach

The potential energy of the system, Π , is given by

$$\Pi = \frac{1}{2} \int_A \boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon} t dA - \int_A \mathbf{u}^T \mathbf{f} t dA - \int_L \mathbf{u}^T \mathbf{T} t d\ell - \sum_i \mathbf{u}^T \mathbf{P}_i \quad (5.27)$$

In the last term above, i indicates the point of application of a point load \mathbf{P}_i and $\mathbf{P}_i = [P_x, P_y]^T$. The summation on i gives the potential energy due to all point loads.

Using the triangulation shown in Fig. 5.2, the total potential energy can be written in the form

$$\Pi = \sum_e \frac{1}{2} \int_e \boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon} t dA - \sum_e \int_e \mathbf{u}^T \mathbf{f} t dA - \int_L \mathbf{u}^T \mathbf{T} t d\ell - \sum_i \mathbf{u}^T \mathbf{P}_i \quad (5.28a)$$

or

$$\Pi = \sum_e U_e - \sum_e \int_e \mathbf{u}^T \mathbf{f} t dA - \int_L \mathbf{u}^T \mathbf{T} t d\ell - \sum_i \mathbf{u}^T \mathbf{P}_i \quad (5.28b)$$

where $U_e = \frac{1}{2} \int_e \boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon} t dA$ is the element strain energy.

Element Stiffness

We now substitute for the strain from the element strain–displacement relationship in Eq. 5.25 into the element strain energy U_e in Eq. 5.28b, to obtain

$$\begin{aligned} U_e &= \frac{1}{2} \int_e \boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon} t dA \\ &= \frac{1}{2} \int_e \mathbf{q}^T \mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{q} t dA \end{aligned} \quad (5.29a)$$

Taking the element thickness t_e as constant over the element, and since all terms in the \mathbf{D} and \mathbf{B} matrices are constants, we have

$$U_e = \frac{1}{2} \mathbf{q}^T \mathbf{B}^T \mathbf{D} \mathbf{B} t_e \left(\int_e dA \right) \mathbf{q} \quad (5.29b)$$

Now, $\int_e dA = A_e$, where A_e is the area of the element. Thus,

$$U_e = \frac{1}{2} \mathbf{q}^T t_e A_e \mathbf{B}^T \mathbf{D} \mathbf{B} \mathbf{q} \quad (5.29c)$$

or

$$U_e = \frac{1}{2} \mathbf{q}^T \mathbf{k}^e \mathbf{q} \quad (5.29d)$$

where \mathbf{k}^e is the element stiffness matrix given by

$$\mathbf{k}^e = t_e A_e \mathbf{B}^T \mathbf{D} \mathbf{B} \quad (5.30)$$

For plane stress or plane strain, the element stiffness matrix can be obtained by taking the appropriate material property matrix \mathbf{D} defined in Chapter 1, and carrying out the above multiplication on the computer. We note that \mathbf{k}^e is symmetric since \mathbf{D} is symmetric. The element connectivity as established in Table 5.1 is now used to add the element stiffness values in \mathbf{k}^e into the corresponding global locations in the global stiffness matrix \mathbf{K} , so that

$$\begin{aligned} U &= \sum_e \frac{1}{2} \mathbf{q}^T \mathbf{k}^e \mathbf{q} \\ &= \frac{1}{2} \mathbf{Q}^T \mathbf{K} \mathbf{Q} \end{aligned} \quad (5.31)$$

The global stiffness matrix \mathbf{K} is symmetric and banded or sparse. The stiffness value K_{ij} is zero when the degrees of freedom i and j are not connected through an element. If i and j are connected through one or more elements, stiffness values accumulate from these elements. For the global dof numbering shown in Fig. 5.2, the bandwidth is related to the maximum difference in node numbers of an element, over all the elements. If i_1 , i_2 , and i_3 are node numbers of an element e , the maximum element node number difference is given by

$$m_e = \max (|i_1 - i_2|, |i_2 - i_3|, |i_3 - i_1|) \quad (5.32a)$$

The half-bandwidth is then given by

$$\text{NBW} = 2 \left[\max_{1 \leq e \leq \text{NE}} (m_e) + 1 \right] \quad (5.32b)$$

where NE is the number of elements and 2 is the number of degrees of freedom per node.

The global stiffness \mathbf{K} is in a form where all the degrees of freedom \mathbf{Q} are free. It needs to be modified to account for the boundary conditions.

Force Terms

The **body force term** $\int_e \mathbf{u}^T \mathbf{f} t \, dA$ appearing in the total potential energy in Eq. 5.28b is considered first. We have,

$$\int_e \mathbf{u}^T \mathbf{f} t \, dA = t_e \int_e (u f_x + v f_y) \, dA$$

Using the interpolation relations given in Eq. 5.12a, we find

$$\int_e \mathbf{u}^T \mathbf{f} t \, dA = q_1 \left(t_e f_x \int_e N_1 \, dA \right) + q_2 \left(t_e f_y \int_e N_1 \, dA \right)$$

$$\begin{aligned}
 &+ q_3 \left(t_e f_x \int_e N_2 dA \right) + q_4 \left(t_e f_y \int_e N_2 dA \right) \\
 &+ q_5 \left(t_e f_x \int_e N_3 dA \right) + q_6 \left(t_e f_y \int_e N_3 dA \right)
 \end{aligned} \quad (5.33)$$

From the definition of shape functions on a triangle, shown in Fig 5.4, $\int_e N_1 dA$ represents the volume of a tetrahedron with base area A_e and height of corner equal to 1 (nondimensional). The volume of this tetrahedron is given by $\frac{1}{3} \times \text{Base area} \times \text{Height}$ (Fig. 5.6).

$$\int_e N_1 dA = \frac{1}{3} A_e \quad (5.34)$$

Similarly, $\int_e N_2 dA = \int_e N_3 dA = \frac{1}{3} A_e$. Equation 5.33 can now be written in the form

$$\int_e \mathbf{u}^T \mathbf{f}^e dA = \mathbf{q}^T \mathbf{f}^e \quad (5.35)$$

where \mathbf{f}^e is the element body force vector, given as

$$\mathbf{f}^e = \frac{t_e A_e}{3} [f_x, f_y, f_x, f_y, f_x, f_y]^T \quad (5.36)$$

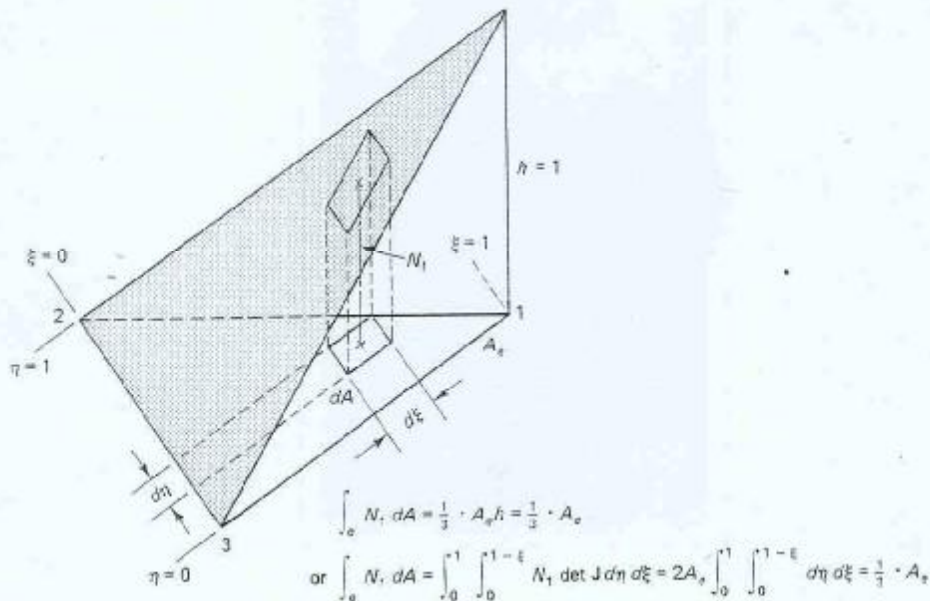


Figure 5.6 Integral of a shape function.

These element nodal forces contribute to the global load vector \mathbf{F} . The connectivity in Table 5.1 needs to be used again to add \mathbf{f}^e to the global force vector \mathbf{F} . The vector \mathbf{f}^e is of dimension (6×1) , whereas \mathbf{F} is $(N \times 1)$. This assembly procedure is discussed in Chapters 3 and 4. Stating this symbolically,

$$\mathbf{F} \leftarrow \sum_e \mathbf{f}^e \quad (5.37)$$

A **traction force** is a distributed load acting on the surface of the body. Such a force acts on edges connecting boundary nodes. A traction force acting on the edge of an element contributes to the global load vector \mathbf{F} . This contribution can be determined by considering the traction force term $\int \mathbf{u}^T \mathbf{T} t \, d\ell$. Consider an edge ℓ_{1-2} , acted on by a traction T_x, T_y in units of force per unit surface area, shown in Fig. 5.7. We have

$$\int_{\ell} \mathbf{u}^T \mathbf{T} t \, d\ell = \int_{\ell_{1-2}} (uT_x + vT_y) t \, d\ell \quad (5.38a)$$

Using $\mathbf{u} = \mathbf{N}\mathbf{q}$, we get

$$\begin{aligned} \int_{\ell_{1-2}} \mathbf{u}^T \mathbf{T} t \, d\ell &= q_1 \left(t_e T_x \int N_1 \, d\ell \right) + q_2 \left(t_e T_y \int N_1 \, d\ell \right) \\ &+ q_3 \left(t_e T_x \int N_2 \, d\ell \right) + q_4 \left(t_e T_y \int N_2 \, d\ell \right) \end{aligned} \quad (5.38b)$$

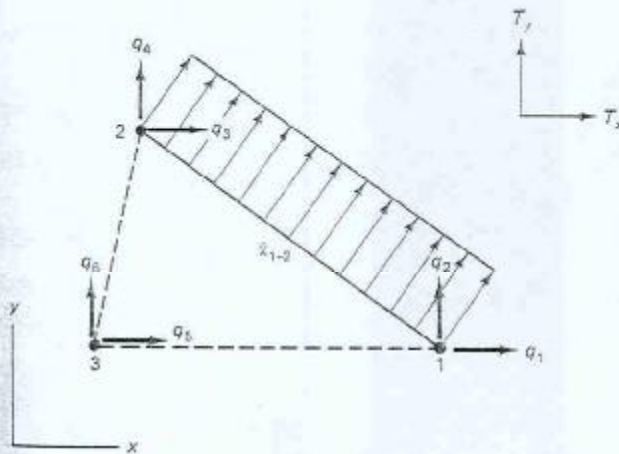


Figure 5.7 Traction load.

We note here that N_3 is zero along the edge 1-2, and N_1 and N_2 are similar to the shape functions in one dimension satisfying $N_1 + N_2 = 1$. Each of the integrals in Eq. 5.38b above equals one-half the base length ℓ_e times the height ($=1$):

$$\int_{\ell_{1-2}} N_i \, d\ell = \frac{1}{2} \ell_{1-2} \quad (5.39)$$

where

$$\ell_{1-2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The traction term is now given by

$$\int_{\ell_{1-2}} \mathbf{u}^T \mathbf{T} t \, d\ell = \bar{\mathbf{q}}^T \mathbf{T}^e \quad (5.40)$$

where $\bar{\mathbf{q}}$ is the set of nodal degrees of freedom corresponding to the edge 1-2. That is,

$$\bar{\mathbf{q}} = [q_1, q_2, q_3, q_4]^T \quad (5.41)$$

and

$$\mathbf{T}^e = \frac{t_e \ell_{1-2}}{2} [T_x, T_y, T_x, T_y]^T \quad (5.42)$$

The traction load contributions need to be added to the global force vector \mathbf{F} .

The **point load** term is easily considered by having a node at the point of application of the point load. If i is the node at which $\mathbf{P}_i = [P_x, P_y]^T$ is applied, then

$$\mathbf{u}^T \mathbf{P}_i = Q_{2i-1} P_x + Q_{2i} P_y \quad (5.43)$$

Thus, P_x and P_y , the x and y components of \mathbf{P}_i , get added to the $(2i - 1)$ th and $(2i)$ th components of the global force \mathbf{F} .

The contribution of body forces, traction forces, and point loads to the global force \mathbf{F} can be represented as $\mathbf{F} \leftarrow \sum_e (\mathbf{f}^e + \mathbf{T}^e) + \mathbf{P}$.

Consideration of the strain energy and the force terms gives us the total potential energy in the form

$$\Pi = \frac{1}{2} \mathbf{Q}^T \mathbf{K} \mathbf{Q} - \mathbf{Q}^T \mathbf{F} \quad (5.44)$$

The stiffness and force modifications are made to account for the boundary conditions. Using the methods presented in Chapters 3 and 4, we have

$$\mathbf{K} \mathbf{Q} = \mathbf{F} \quad (5.45)$$

where \mathbf{K} and \mathbf{F} are modified stiffness matrix and force vector, respectively. These equations are solved by Gaussian elimination or other techniques, to yield the displacement vector \mathbf{Q} .

Galerkin Approach

Following the steps presented in Chapter 1, we introduce

$$\boldsymbol{\phi} = [\phi_x, \phi_y]^T \quad (5.46)$$

and

$$\boldsymbol{\epsilon}(\boldsymbol{\phi}) = \left[\frac{\partial \phi_x}{\partial x}, \frac{\partial \phi_y}{\partial y}, \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right]^T \quad (5.47)$$

where ϕ is an arbitrary (virtual) displacement vector, consistent with the boundary conditions. The variational form is given by

$$\int_A \sigma^T \epsilon(\phi) t dA - \left(\int_A \phi^T f t dA + \int_L \phi^T T t d\ell + \sum_i \phi_i^T P_i \right) = 0 \quad (5.48)$$

where the first term represents the internal virtual work. The expression in parentheses represents the external virtual work. On the discretized region, the above equation becomes

$$\sum_e \int_e \epsilon^T D \epsilon(\phi) t dA - \left(\sum_e \int_e \phi^T f t dA + \int_L \phi^T T t d\ell + \sum_i \phi_i^T P_i \right) = 0 \quad (5.49)$$

Using the interpolation steps of Eqs. 5.12–5.14, we express

$$\phi = N \psi \quad (5.50)$$

$$\epsilon(\phi) = B \psi \quad (5.51)$$

where

$$\psi = [\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6]^T \quad (5.52)$$

represents the arbitrary nodal displacements of element e . The global nodal displacement variations Ψ are represented by

$$\Psi = [\Psi_1, \Psi_2, \dots, \Psi_N]^T \quad (5.53)$$

The element internal work term in Eq. 5.49 can be expressed as

$$\int_e \epsilon^T D \epsilon(\phi) t dA = \int_e q^T B^T D B \psi t dA$$

Noting that all terms of B and D are constant, and denoting t_e and A_e as thickness and area of element, respectively, we find

$$\begin{aligned} \int_e \epsilon^T D \epsilon(\phi) t dA &= q^T B^T D B t_e \int_e dA \psi \\ &= q^T t_e A_e B^T D B \psi \\ &= q^T k^e \psi \end{aligned} \quad (5.54)$$

where k^e is the element stiffness matrix given by

$$k^e = t_e A_e B^T D B \quad (5.55)$$

The material property matrix D is symmetric, and, hence, the element stiffness matrix is also symmetric. The element connectivity as presented in Table 5.1 is used in adding the stiffness values of k^e to the global locations. Thus,

$$\begin{aligned} \sum_e \int_e \boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon}(\boldsymbol{\phi}) t \, dA &= \sum_e \mathbf{q}^T \mathbf{k}^e \boldsymbol{\psi} = \sum_e \boldsymbol{\psi}^T \mathbf{k}^e \mathbf{q} \\ &= \boldsymbol{\Psi}^T \mathbf{K} \mathbf{Q} \end{aligned} \quad (5.56)$$

The global stiffness matrix \mathbf{K} is symmetric and banded. The treatment of external virtual work terms follows the steps involved in the treatment of force terms in the potential energy formulation, where \mathbf{u} is replaced by $\boldsymbol{\phi}$. Thus,

$$\int_e \boldsymbol{\phi}^T \mathbf{f} t \, dA = \boldsymbol{\psi}^T \mathbf{f}^e \quad (5.57)$$

which follows from Eq. 5.33, with \mathbf{f}^e given by Eq. 5.36. Similarly, the traction and point load treatment follows from Eqs. 5.38 and 5.43. The terms in the variational form are given by

$$\text{Internal virtual work} = \boldsymbol{\Psi}^T \mathbf{K} \mathbf{Q} \quad (5.58a)$$

$$\text{External virtual work} = \boldsymbol{\Psi}^T \mathbf{F} \quad (5.58b)$$

The stiffness and force matrices are modified to use the full size (all degrees of freedom), using methods suggested in Chapter 3. From the Galerkin form (Eq. 5.49), the arbitrariness of $\boldsymbol{\Psi}$ gives

$$\mathbf{K} \mathbf{Q} = \mathbf{F} \quad (5.59)$$

where \mathbf{K} and \mathbf{F} are modified to account for boundary conditions. Equation 5.59 turns out to be the same as Eq. 5.45, obtained in the potential energy formulation.

Stress Calculations

Since strains are constant in a constant strain triangle (CST) element, the corresponding stresses are constant. The stress values need to be calculated for each element. Using the stress-strain relations in Eq. 5.6 and element strain-displacement relations in Eq. 5.25, we have

$$\boldsymbol{\sigma} = \mathbf{D} \mathbf{B} \mathbf{q} \quad (5.60)$$

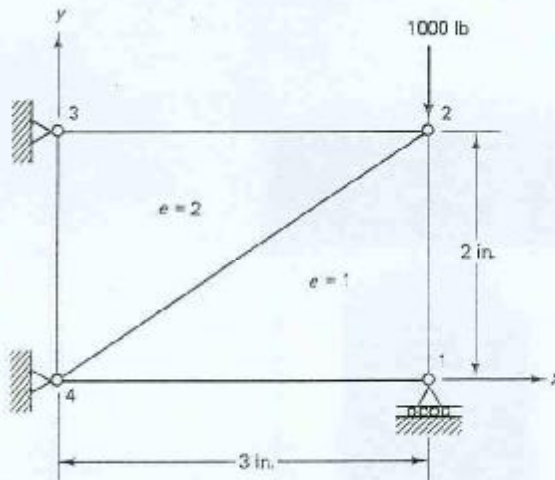
The connectivity in Table 5.1 is once again needed to extract the element nodal displacements \mathbf{q} from the global displacements vector \mathbf{Q} . Equation 5.60 is used to calculate the element stresses. For interpolation purposes, the calculated stress may be used as the value at the centroid of the element.

Principal stresses and their directions are calculated using Mohr's circle relationships. The program at the end of the chapter includes the principal stress calculations.

Detailed calculations in the example below illustrate the steps involved. However, it is expected that the exercise problems at the end of the chapter will be solved using a computer.

Example 5.4

For the two-dimensional loaded plate shown in Fig. E5.4, determine the displacements of nodes 1 and 2 and the element stresses using plane stress conditions. Body force may be neglected in comparison to the external forces.



Thickness $t = 0.5$ in., $E = 30 \times 10^6$ psi, $\nu = 0.25$

Figure E5.4

Solution For plane stress conditions, the material property matrix is given by

$$D = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} = \begin{bmatrix} 3.2 \times 10^7 & 0.8 \times 10^7 & 0 \\ 0.8 \times 10^7 & 3.2 \times 10^7 & 0 \\ 0 & 0 & 1.2 \times 10^7 \end{bmatrix}$$

Using the local numbering pattern used in Fig. E5.3, we establish the connectivity as follows:

| Element No. | Nodes | | |
|-------------|-------|---|---|
| | 1 | 2 | 3 |
| 1 | 1 | 2 | 4 |
| 2 | 3 | 4 | 2 |

On performing the matrix multiplication DB^T , we get

$$DB^T = 10^7 \begin{bmatrix} 1.067 & -0.4 & 0 & 0.4 & -1.067 & 0 \\ 0.267 & -1.6 & 0 & 1.6 & -0.267 & 0 \\ -0.6 & 0.4 & 0.6 & 0 & 0 & -0.4 \end{bmatrix}$$

and

$$\mathbf{DB}^e = 10^7 \begin{bmatrix} -1.067 & 0.4 & 0 & -0.4 & 1.067 & 0 \\ -0.267 & 1.6 & 0 & -1.6 & 0.267 & 0 \\ 0.6 & -0.4 & -0.6 & 0 & 0 & 0.4 \end{bmatrix}$$

These two relationships will be used later in calculating stresses using $\sigma^e = \mathbf{DB}^e \mathbf{q}$. The multiplication $t_e A_e \mathbf{B}^{eT} \mathbf{DB}^e$ gives the element stiffness matrices,

$$\mathbf{k}^1 = 10^7 \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & \text{Symmetric} & & & 0.533 & 0 \\ & & & & & 0.2 \end{bmatrix} \begin{matrix} \text{Global} \\ \leftarrow \text{dof} \end{matrix}$$

$$\mathbf{k}^2 = 10^7 \begin{bmatrix} 5 & 6 & 7 & 8 & 3 & 4 \\ 0.983 & -0.5 & -0.45 & 0.2 & -0.533 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & \text{Symmetric} & & & 0.533 & 0 \\ & & & & & 0.2 \end{bmatrix} \begin{matrix} \text{Global} \\ \leftarrow \text{dof} \end{matrix}$$

In the above element matrices, the global dof association is shown on top. In the problem under consideration, Q_2 , Q_5 , Q_6 , Q_7 , and Q_8 are all zero. Using the elimination approach discussed in Chapter 3, it is now sufficient to consider the stiffnesses associated with the degrees of freedom Q_1 , Q_3 , Q_4 . Since the body forces are neglected, the force vector has the component $F_4 = -1000$ lb. The set of equations is given by the matrix representation

$$10^7 \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1000 \end{Bmatrix}$$

Solving for Q_1 , Q_3 , and Q_4 , we get

$$Q_1 = 1.913 \times 10^{-5} \text{ in.} \quad Q_3 = 0.875 \times 10^{-5} \text{ in.} \quad Q_4 = -7.436 \times 10^{-5} \text{ in.}$$

For element 1, the element nodal displacement vector is given by

$$\mathbf{q}^1 = 10^{-5} [1.913, 0, 0.875, -7.436, 0, 0]^T$$

The element stresses σ^1 are calculated from $\mathbf{DB}^1 \mathbf{q}$ as

$$\sigma^1 = [-93.3, -1138.7, -62.3]^T \text{ psi}$$

Similarly

$$\mathbf{q}^e = 10^{-5}[0, 0, 0, 0, 0.875, -7.436]^T$$

$$\boldsymbol{\sigma}^e = [93.4, 23.4, -297.4]^T \text{ psi}$$

The computer results may differ slightly since the penalty approach for handling boundary conditions is used in the computer program. ■

Temperature Effects

If the distribution of the change in temperature $\Delta T(x, y)$ is known, the strain due to this change in temperature can be treated as an initial strain $\boldsymbol{\epsilon}_0$. From the theory of mechanics of solids, $\boldsymbol{\epsilon}_0$ can be represented by

$$\boldsymbol{\epsilon}_0 = [\alpha \Delta T, \alpha \Delta T, 0]^T \quad (5.61)$$

for plane stress, and

$$\boldsymbol{\epsilon}_0 = (1 + \nu)[\alpha \Delta T, \alpha \Delta T, 0]^T \quad (5.62)$$

for plane strain. The stresses and strains are related by

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) \quad (5.63)$$

The effect of temperature can be accounted for by considering the strain energy term. We have

$$\begin{aligned} U &= \frac{1}{2} \int (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0)^T \mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0) t \, dA \\ &= \frac{1}{2} \int (\boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon} - 2\boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon}_0 + \boldsymbol{\epsilon}_0^T \mathbf{D} \boldsymbol{\epsilon}_0) t \, dA \end{aligned} \quad (5.64)$$

The first term in the expansion above gives the stiffness matrix derived before. The last term is a constant, which has no effect on the minimization process. The middle term, which yields the temperature load, is now considered in detail. Using the strain-displacement relationship $\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q}$,

$$\int_A \boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon}_0 t \, dA = \sum_e \mathbf{q}^T (\mathbf{B}^T \mathbf{D} \boldsymbol{\epsilon}_0) t_e A_e \quad (5.65)$$

This step is directly obtained in the Galerkin approach where $\boldsymbol{\epsilon}^T$ will be $\boldsymbol{\epsilon}^T(\phi)$ and \mathbf{q}^T will be $\boldsymbol{\psi}^T$.

It is convenient to designate the element temperature load as

$$\boldsymbol{\Theta}^e = t_e A_e \mathbf{B}^T \mathbf{D} \boldsymbol{\epsilon}_0 \quad (5.66)$$

where

$$\boldsymbol{\Theta}^e = [\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6]^T \quad (5.67)$$

The vector ϵ_0 is the strain in Eq. 5.61 or 5.62 due to the average temperature change in the element. Θ^e represents the element nodal load contributions that must be added to the global force vector using the connectivity.

The stresses in an element are then obtained by using Eq. 5.63 in the form

$$\sigma = D(Bq - \epsilon_0) \quad (5.68)$$

5.4 PROBLEM MODELING AND BOUNDARY CONDITIONS

The finite element method is used for computing displacements and stresses for a wide variety of problems. The physical dimensions, loading, and boundary conditions are clearly defined in some problems, similar to what we discussed in Example 5.4. In other problems, these are not clear at the outset.

An example is the problem illustrated in Fig. 5.8a. A plate with such a loading can exist anywhere in space. Since we are interested in the deformation of the body, the symmetry of the geometry and the symmetry of the loading can be used effectively. Let x and y represent the axes of symmetry as shown in Fig. 5.8b. The points along the x axis move along x and are constrained in the y direction and points along the y axis are constrained along the x direction. This suggests that the part, which is one-quarter of the full area, with the loading and boundary conditions as shown is all that is needed to solve for the deformation and stresses.

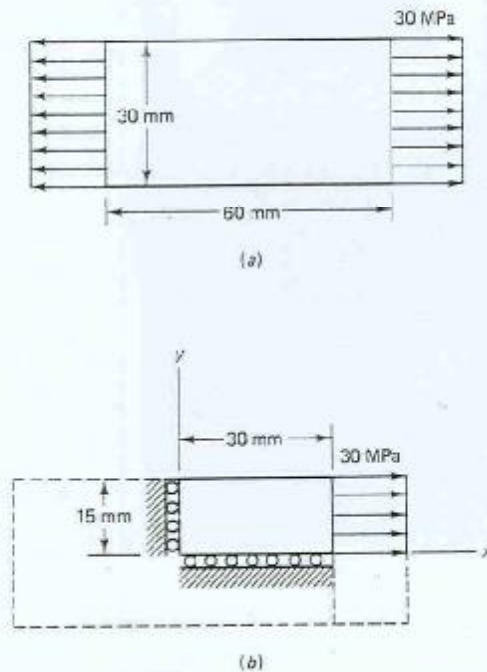


Figure 5.8 Rectangular plate.

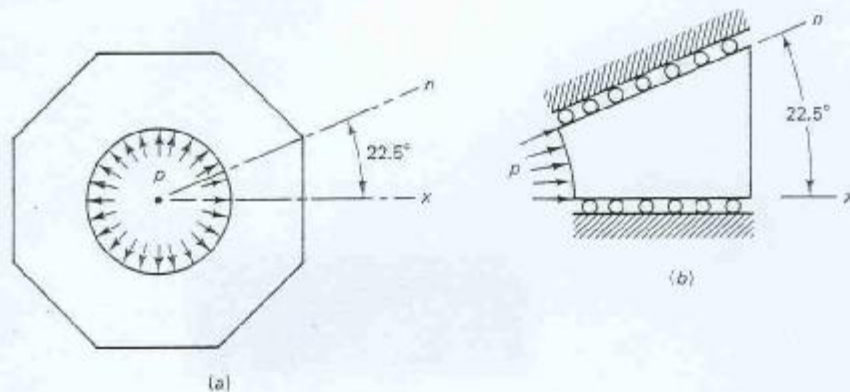


Figure 5.9 Octagonal pipe.

As another example, consider an octagonal pipe under internal pressure, shown in Fig. 5.9a. By symmetry, we observe that it is sufficient to consider the 22.5° segment shown in Fig. 5.9b. The boundary conditions require that points along x and n are constrained normal to the two lines, respectively. Note that for a *circular* pipe under internal or external pressure, by symmetry, all points move radially. In this case, any radial segment may be considered. The boundary conditions for points along the x axis in Fig. 5.9b are easily considered by using the penalty approach discussed in Chapter 3. The boundary conditions for points along the inclined direction n , which are considered perpendicular to n , are now treated in detail. If node i with degrees of freedom Q_{2i-1} and Q_{2i} moves along n as seen in Fig. 5.10, and θ is the angle of inclination of n with respect to x axis, we have

$$Q_{2i-1} \sin \theta - Q_{2i} \cos \theta = 0 \quad (5.69)$$

This boundary condition is seen to be a multipoint constraint, which is discussed in Chapter 3. Using the penalty approach presented in Chapter 3, this amounts to adding a term to the potential energy:

$$\Pi = \frac{1}{2} \mathbf{Q}^T \mathbf{K} \mathbf{Q} - \mathbf{Q}^T \mathbf{F} + \frac{1}{2} C (Q_{2i-1} \sin \theta - Q_{2i} \cos \theta)^2 \quad (5.70)$$

where C is a large number.

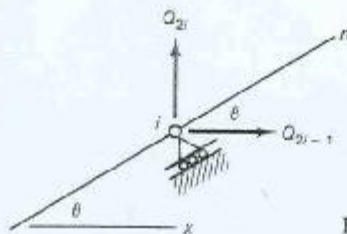


Figure 5.10 Inclined roller support.

The squared term in Eq. 5.70 can be written in the form

$$\frac{1}{2}C(Q_{2i-1}\sin\theta - Q_{2i}\cos\theta)^2 = \frac{1}{2}[Q_{2i-1}, Q_{2i}] \begin{bmatrix} C\sin^2\theta & -C\sin\theta\cos\theta \\ -C\sin\theta\cos\theta & C\cos^2\theta \end{bmatrix} \begin{Bmatrix} Q_{2i-1} \\ Q_{2i} \end{Bmatrix} \quad (5.71)$$

The terms $C\sin^2\theta$, $-C\sin\theta\cos\theta$, and $C\cos^2\theta$ get added to the global stiffness matrix, for every node on the incline, and the new stiffness matrix is used to solve for the displacements. Note that the above modifications can also be directly obtained from Eq. 3.82 by substituting $\beta_0 = 0$, $\beta_1 = \sin\theta$, and $\beta_2 = -\cos\theta$. The contributions to the banded stiffness matrix S are made in the locations $(2i - 1, 1)$, $(2i - 1, 2)$, and $(2i, 1)$ by adding $C\sin^2\theta$, $-C\sin\theta\cos\theta$, $C\cos^2\theta$, respectively.

Some General Comments on Dividing into Elements

When dividing an area into triangles, avoid large aspect ratios. Aspect ratio is defined as the ratio of maximum to minimum characteristic dimensions. Observe that the best elements are those that approach an equilateral triangular configuration. Such configurations are not usually possible. A good practice may be to choose corner angles in the range of 30° to 120° .

In problems where the stresses change widely over an area, such as in notches and fillets, it is good practice to decrease the size of elements in that area to capture the stress variations. The constant strain triangle (CST), in particular, gives constant stresses on the element. This suggests that smaller elements will better represent the distribution. Better estimates of maximum stress may be obtained even with coarser meshes by plotting and extrapolating. For this purpose, the constant element stresses may be interpreted as the values at centroids of the triangle. A method for evaluating nodal values from constant element values is presented in the postprocessing section of Chapter 12.

Coarse meshes are recommended for initial trials to check data and reasonableness of results. Errors may be fixed at this stage, before running larger numbers of elements. Increasing the number of elements in those regions where stress variations are high should give better results. This is called *convergence*. One should get a feel for convergence by successively increasing the number of elements in finite element meshes.

Example 5.5

The solution of Example 5.4 using program FE2CST is presented below, using interactive mode of input.

```
Number of Elements =? 2
Number of Nodes =? 4
Number of Constrained DOF =? 5
Number of Component Loads =? 1
```


PROBLEMS

- 5.1. The nodal coordinates of the triangular element are shown in Fig. P5.1. At the interior point P , the x -coordinate is 3.3 and $N_1 = 0.5$. Determine N_2 , N_3 , and the y -coordinate at point P .

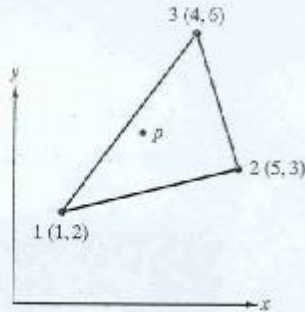


FIGURE P5.1

- 5.2. Determine the Jacobian for the $(x, y) - (\xi, \eta)$ transformation for the element shown in Fig. P5.2. Also, find the area of the triangle.

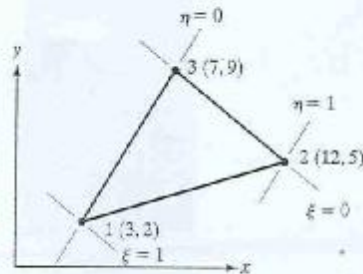


FIGURE P5.2

- 5.3. For point P located inside the triangle shown in Fig. P5.3, the shape functions N_1 and N_2 are 0.15 and 0.25, respectively. Determine the x - and y -coordinates of point P .

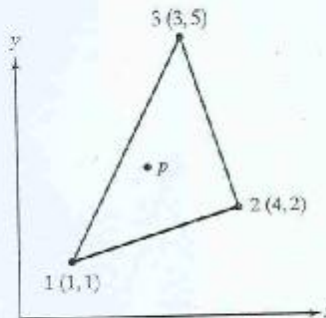


FIGURE P5.3

ANGLE SX->S1
+03 -
-4.1642E+01

1. At the interior
the y-coordinate

ement shown in

- 5.4. In Example 5.1, determine the shape functions using the area coordinate approach. (Hint: Use Area = $0.5(x_1y_2 - x_2y_1)$ for triangle 1-2-3.)
- 5.5. For the triangular element shown in Figure P5.5, obtain the strain-displacement relation matrix **B** and determine the strains ϵ_x , ϵ_y , and γ_{xy} .

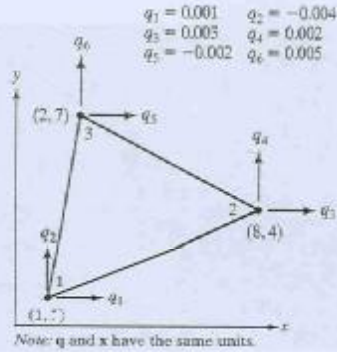


FIGURE P5.5

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- 5.6. Figure P5.6 shows a 2-D region modeled with 12 CST elements.
- (a) Determine the bandwidth NBW (also referred to as the "half-bandwidth").
- (b) If the multipoint constraint $Q_1 = Q_{18}$ is imposed (1 and 18 are degree of freedom numbers corresponding to x displacement of node 1 and y displacement of node 9, respectively), what is the new value of NBW?

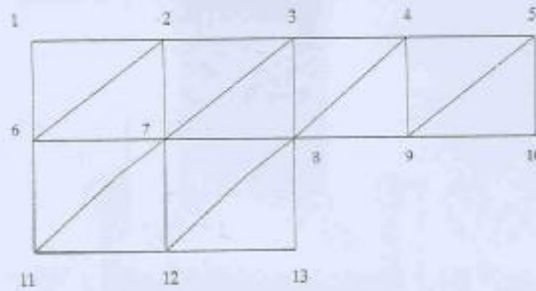
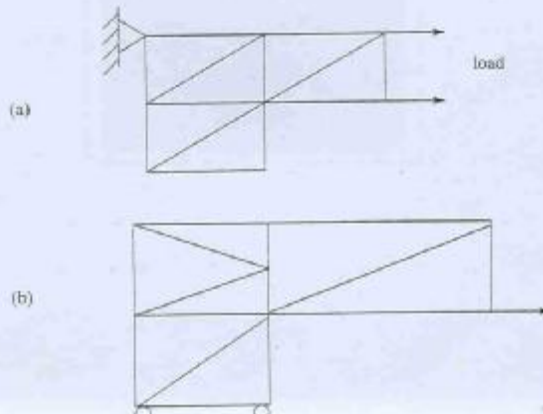


FIGURE P5.6

- 5.7. Indicate all the mistakes in the following finite element models with CST elements:



- 5.8. For a two-dimensional triangular element, the stress-displacement matrix \mathbf{DB} appearing in $\sigma = \mathbf{DBq}$ is given by

$$\mathbf{DB} = \begin{bmatrix} 2500 & 2200 & -1500 & 1200 & -4400 & 1000 \\ 5500 & 4000 & 4100 & 2600 & -1500 & 1200 \\ 2000 & 2500 & -4000 & 1800 & 2200 & 4400 \end{bmatrix} \text{ N/mm}^3$$

If the coefficient of linear expansion is $10 \times 10^{-6}/^\circ\text{C}$, the temperature rise of the element is 100°C , and the volume of the element is 25 mm^3 , determine the equivalent temperature load θ for the element.

- 5.9. For the configuration shown in Fig. P5.9, determine the deflection at the point of load application using a one-element model. If a mesh of several triangular elements is used, comment on the stress values in the elements close to the tip.

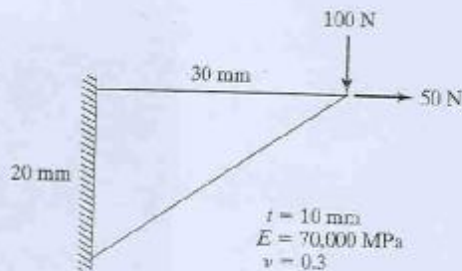


FIGURE P5.9

- 5.10. Determine the bandwidth for the two-dimensional region for the triangular element division with the node numbering shown in Fig. P5.10. How do you proceed to decrease the bandwidth?

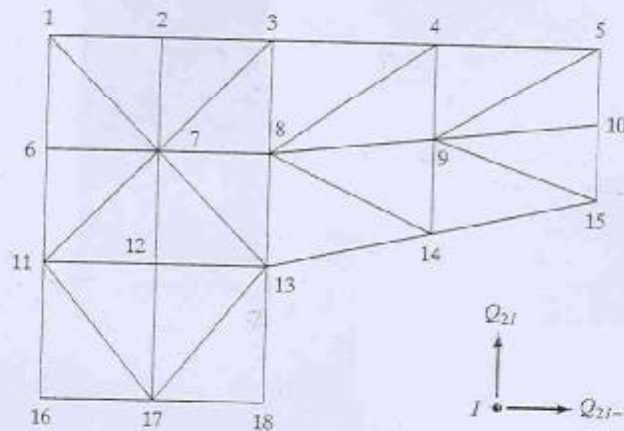


FIGURE P5.10

5.11. Consider the four-element CST model in Fig. P5.11 subjected to a body force $f = y^2 \text{ N/m}^3$ in the y direction. Assemble the global load vector $F_{12 \times 1}$ for the model.

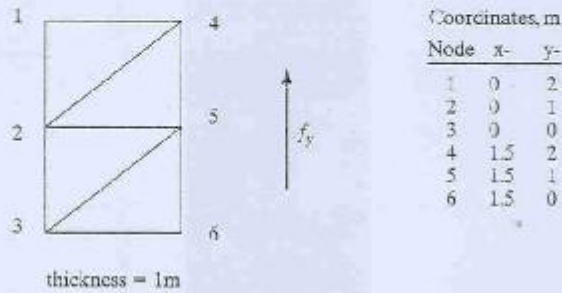


FIGURE P5.11

5.12. Assemble the load vector $F_{6 \times 1}$ at the three nodes on the inner boundary, which is subjected to a pressure $p = 0.9 \text{ MPa}$. (See Fig. P5.12.)

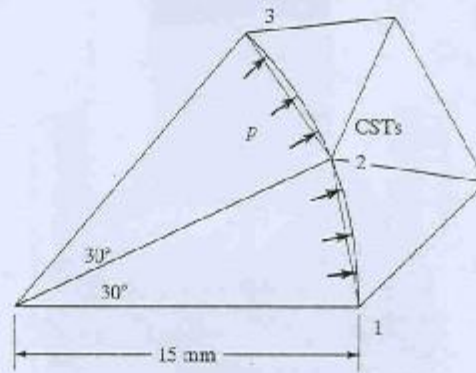


FIGURE P5.12

5.13. Consider the three-noded triangular element in Fig. P5.13. Express the integral for area moment of inertia $I = \int_C y^2 dA$ as

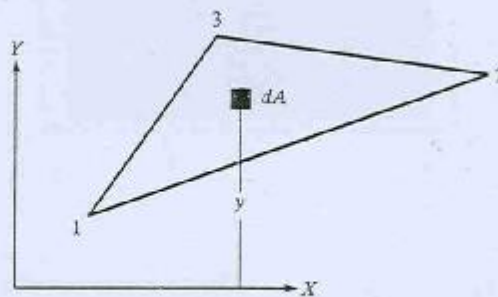


FIGURE P5.13

$$I = \mathbf{y}_c^T \mathbf{R} \mathbf{y}_c$$

where $\mathbf{y}_c = [y_1, y_2, y_3]^T$ = a vector of y -coordinates of the three nodes, and \mathbf{R} is a 3×3 matrix. (*Hint:* Interpolate y using shape functions N_i .)

- 5.14. Compute the integral $I = \int_e N_1 N_2 N_3 dA$, where N_i are the linear shape functions for a three-noded CST element.
- 5.15. Solve the plane stress problem in Fig. P5.15 using three different mesh divisions. Compare your deformation and stress results with values obtained from elementary beam theory.

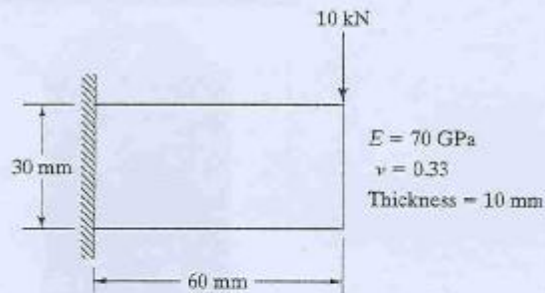


FIGURE P5.15

- 5.16. For the plate with a hole under plane stress (Fig. P5.16), find the deformed shape of the hole and determine the maximum stress distribution along AB by using stresses in elements adjacent to the line. (*Note:* The result in this problem is the same for any thickness. You may use $t = 1$ in.)

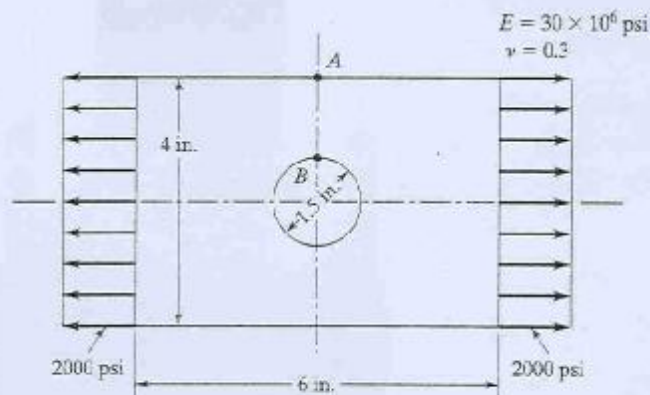


FIGURE P5.16

5.17. Model a half of the disk with a hole (Fig. P5.17) and find the major and minor dimensions after compression. Also, plot the distribution of maximum stress along AB

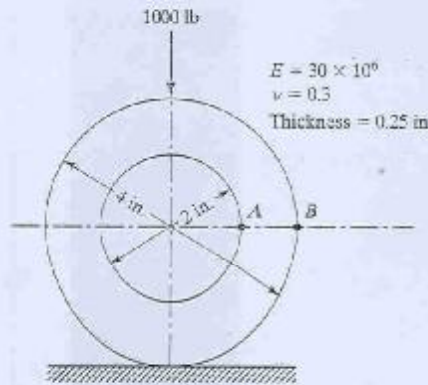


FIGURE P5.17

5.18. Consider the multipoint constraint

$$3Q_5 - 2Q_9 = 0.1$$

where Q_5 is the displacement along degree of freedom 5, and Q_9 is the displacement along dof 9. Write the penalty term

$$\frac{1}{2} C (3Q_5 - 2Q_9 - 0.1)^2 \text{ as } \frac{1}{2} (Q_5, Q_9) \mathbf{k} \begin{pmatrix} Q_5 \\ Q_9 \end{pmatrix} - (Q_5, Q_9) \mathbf{f}$$

and, hence, determine the stiffness additions \mathbf{k} and force additions \mathbf{f} . Then, fill in the following blanks to show how these additions are made in the computer program that uses a banded stiffness matrix \mathbf{S} :

$$S(5, 1) = S(5, 1) + \underline{\hspace{2cm}}$$

$$S(9, 1) = S(9, 1) + \underline{\hspace{2cm}}$$

$$S(5, \underline{\hspace{1cm}}) = S(5, \underline{\hspace{1cm}}) + \underline{\hspace{2cm}}$$

$$F(5) = F(5) + \underline{\hspace{2cm}}$$

$$F(9) = F(9) + \underline{\hspace{2cm}}$$

5.19. Model the 22.5° segment of the octagonal pipe shown in Fig. P5.19. Show the deformed configuration of the segment and the distribution of maximum in-plane shear stress. (*Hint:* For all points along CD , use stiffness modification suggested in Eq. 5.71. Also, maximum in-plane shear stress = $(\sigma_1 - \sigma_2)/2$, where σ_1 and σ_2 are the principal stresses. Assume plane strain.)

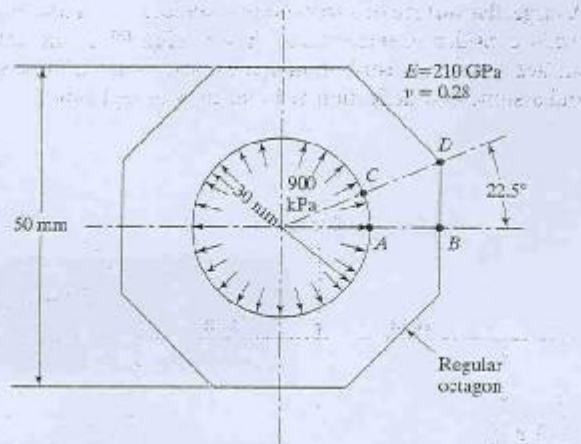


FIGURE P5.19

5.20. Determine the location and magnitude of maximum principal stress and maximum shearing stress in the fillet shown in Fig. P5.20.

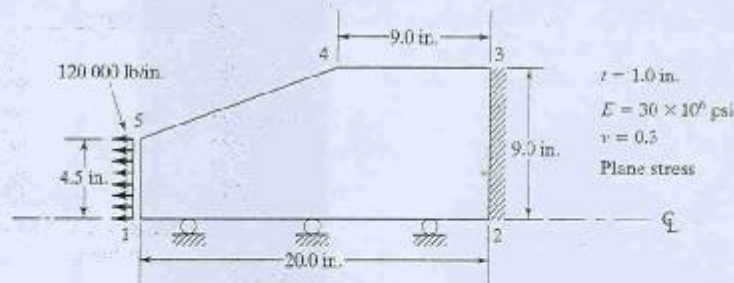
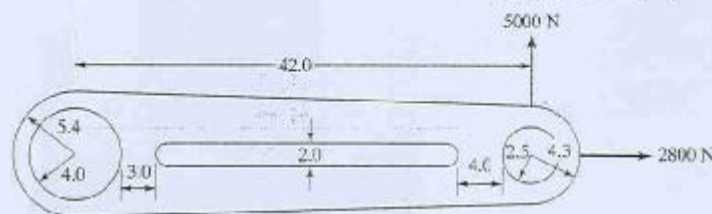


FIGURE P5.20

5.21. The torque arm in Fig. P5.21 is an automotive component. Determine the location and magnitude of maximum von Mises stress, σ_{VM} , given by

$$\sigma_{VM} = \sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2}$$



(All dimensions in cm)

$r = 1.0$ cm
 $E = 200 \times 10^9$ N/m²
 $\nu = 0.3$

FIGURE P5.21

- 5.22. A large, flat surface of a steel body is subjected to a line load of 100 lb/in. Assuming plane strain, consider an enclosure as shown in Fig. P5.22 and determine the deformation of the surface and stress distribution in the body. (Note: Choose small elements close to the load and assume that deflection at 10 in. away is negligible.)

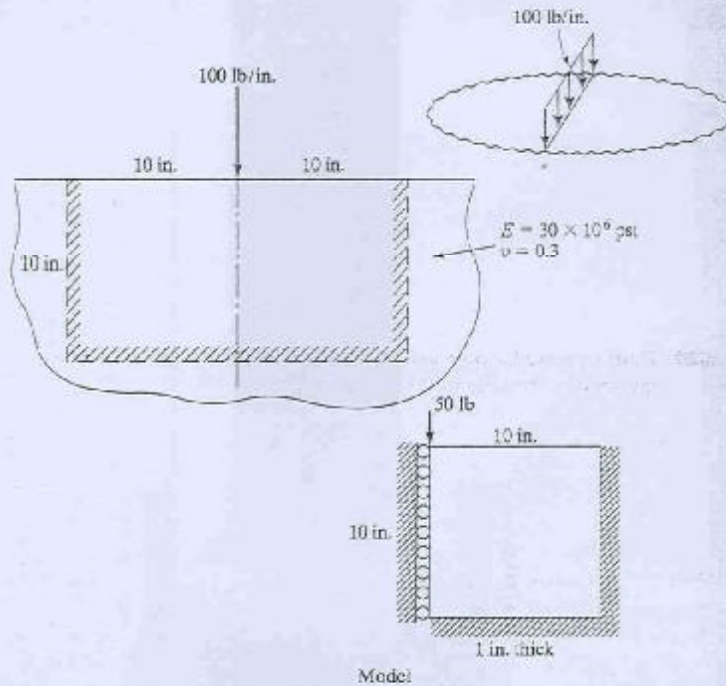


FIGURE P5.22

- 5.23. In Problem 5.22, the load is changed to a distributed load 400 lb/in.² on a 1/4-in.-wide long region, as in Fig. P5.23. Model the problem as above with this loading and find deformation of the surface and stress distribution in the body. (Note: Assume that deflection at 10 in. away is negligible.)

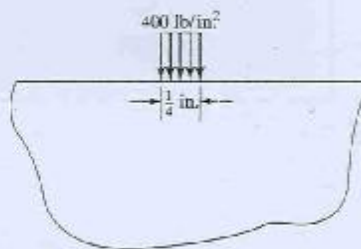


FIGURE P5.23

- 5.24. A $\frac{1}{2} \times 5$ -in. copper piece fits snugly into a short channel-shaped steel piece at room temperature, as shown in Fig. P5.24. The assembly is subjected to a uniform temperature increase of 80°F . Assuming that the properties are constant within this change and that the surfaces are bonded together, find the deformed shape and the stress distribution.

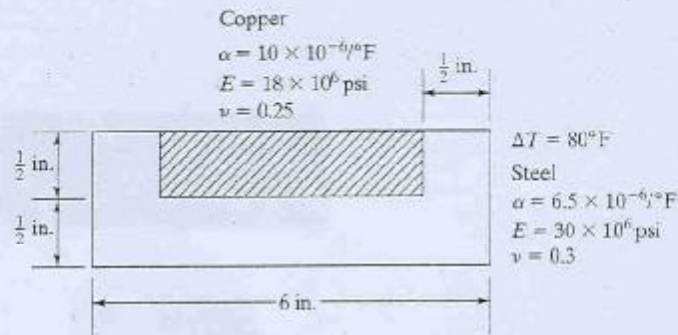


FIGURE P5.24

- 5.25. In the slotted ring shown in Fig. P5.25, two loads of magnitude P and load R are applied such that the 3-mm gap closes. Determine the magnitude of P and show the deformed shape of the part. (*Hint: Find the deflection of gap for, say, $P = 100$ and multiply the deflections proportionately.*)

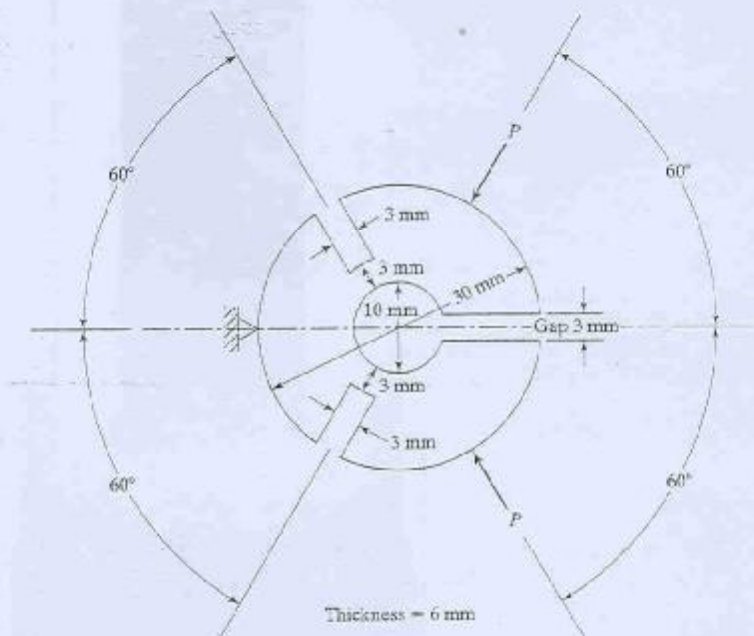


FIGURE P5.25

- 5.26. A titanium piece (A) is press-fitted into a titanium workpiece (B) as shown in Fig. P5.26. Determine the location (show on a sketch) and magnitude of maximum von Mises stress in both the parts (from your CST output file). Then, provide contour plots of the von Mises stress in each part. Data are as follows: $E = 101 \text{ GPa}$; $\nu = 0.34$.

The guidelines are (a) use less than 100 elements in all, (b) mesh each part independently, but without duplicating node or element numbers, (c) choose a value for $L_{\text{interface}}$ and then enforce multipoint constraints (MPCs) between the coincident nodes on this interface—the choice of $L_{\text{interface}}$ will involve trial-and-check as nodes that want to separate should not be forced together through the MPC, and (d) use symmetry. Assume a no-slip interface, a fixed base, and plane strain.

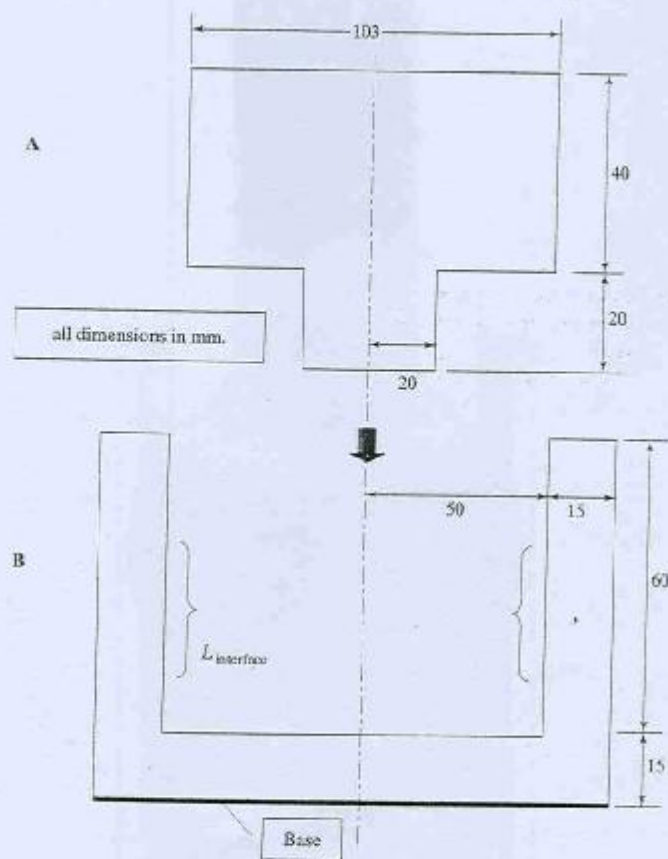


FIGURE P5.26

- 5.27. An edge crack of length a in a rectangular plate is subjected to a tensile stress σ_0 as shown in Fig. P5.27. Using a half-symmetry model, complete the following:
- (a) Determine the crack opening angle, θ ($\theta = 0$ before the load is applied).

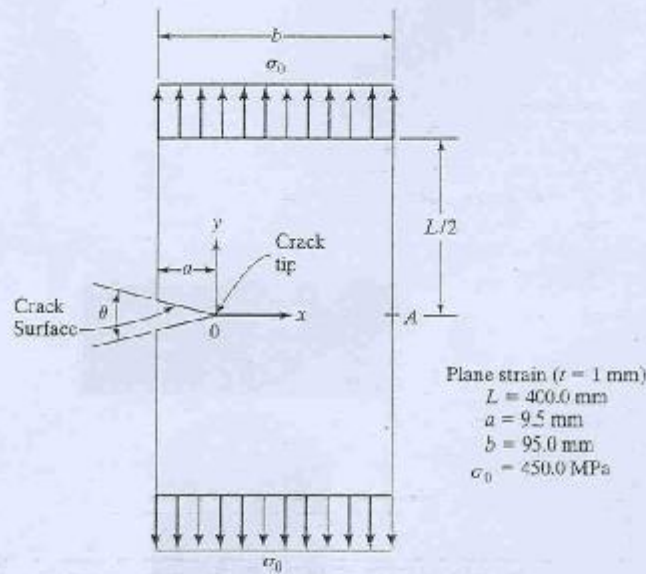


FIGURE P5.27

- (b) Plot the y stress σ_y versus x , along the line $A-O$. Assuming that $\sigma_y = \frac{K_I}{\sqrt{2\pi x}}$, use regression to estimate K_I . Compare your result for infinitely long plates, for which K_I of $= 1.2\sigma_0\sqrt{\pi a}$ is used.
- (c) Repeat part (b) for increasingly fine meshes near the crack tip.
- 5.28. Use the geometry of the plate for the plane-stress problem in P5.15. If the material of the plate is graphite-epoxy resin with fiber orientation at an angle θ to the horizontal, determine the deformation and stress values σ_x , σ_y , and σ_1 , σ_2 for $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ$, and 90° . Properties of graphite in epoxy resin are given in Table 5.1. (*Hint:* The problem solution requires modification of program CST to incorporate the \mathbf{D} matrix defined in Eq. 5.79.)
- 5.29. The plate with a hole in Problem 5.16 is made of pine wood. For $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ$, and 90° , complete the following:
- Determine the deformed shape of the hole.
 - Find the stress distribution along AB and, hence, the stress concentration factor K_I . Plot K_I versus θ .