APPENDIX A

ANALYTIC DETERMINATION OF WEIGHTING COEFFICIENTS

The set of equations to be solved for the determination of the weighting coefficients has the form

$$\sum_{j=1}^{N} w_{ij}\, X_j^{k-1} = (k-1)(k-2)\cdots(k-n)\, X_i^{(k-n-1)}$$

$$i,k = 1,2,\cdots,N \qquad \text{(A-1)}$$

Here, the first-order weighting coefficients with three equally spaced grid points will be calculated analytically for demonstration. For the case considered, N=3, n=1, and i,k = 1,2,3. Then, Eq. (A-1) becomes

$$\sum_{j=1}^{3} w_{ij}\, X_j^{k-1} = (k-1)\, X_i^{(k-2)} \qquad \text{(A-2)}$$

For k = 1, one has from Eq. (A-2)

$$\sum_{j=1}^{3} w_{ij}\, X_j^{0} = 0 \qquad \text{(A-3)}$$

Eq. (A-3) can be expanded as

$$
\begin{aligned}
w_{11} + w_{12} + w_{13} &= 0 \qquad \text{for} \quad i = 1 \\
w_{21} + w_{22} + w_{23} &= 0 \qquad \text{for} \quad i = 2 \\
w_{31} + w_{32} + w_{33} &= 0 \qquad \text{for} \quad i = 3
\end{aligned}
\tag{A-4}
$$

For k = 2, one obtains

$$\sum_{j=1}^{3} w_{ij} X_j = X_i^0 \tag{A-5}$$

In expanded form, this can be rewritten as

$$
\begin{aligned}
w_{11} X_1 + w_{12} X_2 + w_{13} X_3 &= 1 \qquad \text{for} \quad i = 1 \\
w_{21} X_1 + w_{22} X_2 + w_{23} X_3 &= 1 \qquad \text{for} \quad i = 2 \\
w_{31} X_1 + w_{32} X_2 + w_{33} X_3 &= 1 \qquad \text{for} \quad i = 3
\end{aligned}
\tag{A-6}
$$

For k = 3, one gets

$$\sum_{j=1}^{3} w_{ij} X_j^2 = 2 X_i \tag{A-7}$$

Eq. (A-7) can be expanded as

$$
\begin{aligned}
w_{11} X_1^2 + w_{12} X_2^2 + w_{13} X_3^2 &= 2 X_1 \qquad \text{for} \quad i = 1 \\
w_{21} X_1^2 + w_{22} X_2^2 + w_{23} X_3^2 &= 2 X_2 \qquad \text{for} \quad i = 2 \\
w_{31} X_1^2 + w_{32} X_2^2 + w_{33} X_3^2 &= 2 X_3 \qquad \text{for} \quad i = 3
\end{aligned}
\tag{A-8}
$$

In matrix form, Eqs. (A-4), (A-6), and (A-8) can be expressed as

$$\begin{bmatrix} 1 & 1 & 1 \\ X_1 & X_2 & X_3 \\ X_1^2 & X_2^2 & X_3^2 \end{bmatrix} \begin{Bmatrix} w_{11} \\ w_{12} \\ w_{13} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 2X_1 \end{Bmatrix} \quad \text{(A-9)}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ X_1 & X_2 & X_3 \\ X_1^2 & X_2^2 & X_3^2 \end{bmatrix} \begin{Bmatrix} w_{21} \\ w_{22} \\ w_{23} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 2X_2 \end{Bmatrix} \quad \text{(A-10)}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ X_1 & X_2 & X_3 \\ X_1^2 & X_2^2 & X_3^2 \end{bmatrix} \begin{Bmatrix} w_{31} \\ w_{32} \\ w_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 2X_3 \end{Bmatrix} \quad \text{(A-11)}$$

Or, in simple matrix form, this can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ X_1 & X_2 & X_3 \\ X_1^2 & X_2^2 & X_3^2 \end{bmatrix} \begin{Bmatrix} w_{i1} \\ w_{i2} \\ w_{i3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 2X_i \end{Bmatrix} \quad \text{(A-12)}$$

Let

$$\Pi_i(X) \equiv (X-X_1)(X-X_2)\cdots(X-X_{i-1})(X-X_{i+1})\cdots(X-X_N)$$

$$= \sum_{j=1}^{N} C_{ij}\, X^{j-1} \qquad i = 1,2,\ldots,N \qquad \text{(A-13)}$$

where $\Pi_i(X_j) = 0$ for $i \neq j$. Expanding Eq. (A-13) with N=3, one obtains

$$\begin{aligned} C_{11} + C_{12}\, X + C_{13}\, X^2 &= \Pi_1(X) \\ C_{21} + C_{22}\, X + C_{23}\, X^2 &= \Pi_2(X) \\ C_{31} + C_{32}\, X + C_{33}\, X^2 &= \Pi_3(X) \end{aligned} \qquad \text{(A-14)}$$

Substituting X_1, X_2, and X_3 into Eqs. (A-14), and dividing the first equation by $\Pi_1(X_1)$, the second equation by $\Pi_2(X_2)$, and the third equation by $\Pi_3(X_3)$ respectively, one gets in matrix form

$$\begin{bmatrix} \frac{C_{11}}{\Pi_1(X_1)} & \frac{C_{12}}{\Pi_1(X_1)} & \frac{C_{13}}{\Pi_1(X_1)} \\ \frac{C_{21}}{\Pi_2(X_2)} & \frac{C_{22}}{\Pi_2(X_2)} & \frac{C_{23}}{\Pi_2(X_2)} \\ \frac{C_{31}}{\Pi_3(X_3)} & \frac{C_{32}}{\Pi_3(X_3)} & \frac{C_{33}}{\Pi_3(X_3)} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ X_1 & X_2 & X_3 \\ X_1^{\,2} & X_2^{\,2} & X_3^{\,2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(A-15)}$$

One can write Eq. (A-15) as follows:

$$[\,X\,]^{-1}\,[\,X\,] = [\,I\,] \tag{A-16}$$

where $[\,X\,]$ is the Vandermonde matrix and $[\,X\,]^{-1}$ is as follows

$$[\,X\,]^{-1} = \begin{bmatrix} \dfrac{C_{11}}{\Pi_1(X_1)} & \dfrac{C_{12}}{\Pi_1(X_1)} & \dfrac{C_{13}}{\Pi_1(X_1)} \\ \dfrac{C_{21}}{\Pi_2(X_2)} & \dfrac{C_{22}}{\Pi_2(X_2)} & \dfrac{C_{23}}{\Pi_2(X_2)} \\ \dfrac{C_{31}}{\Pi_3(X_3)} & \dfrac{C_{32}}{\Pi_3(X_3)} & \dfrac{C_{33}}{\Pi_3(X_3)} \end{bmatrix} \tag{A-17}$$

In Eq. (A-12), it was shown that

$$\begin{Bmatrix} w_{i1} \\ w_{i2} \\ w_{i3} \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ X_1 & X_2 & X_3 \\ X_1^{\,2} & X_2^{\,2} & X_3^{\,2} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \\ 2X_i \end{Bmatrix} \tag{A-12'}$$

Thus, if one can find $[\,X\,]^{-1}$, the weighting coefficients can be determined.

Note that in Eq. (A-15),

$$\begin{aligned} \Pi_1(X_1) &= (X_1 - X_2)(X_1 - X_3) \\ \Pi_2(X_2) &= (X_2 - X_1)(X_2 - X_3) \\ \Pi_3(X_3) &= (X_3 - X_1)(X_3 - X_2) \end{aligned} \tag{A-18}$$

From Eq. (A-14), one obtains

$$C_{11} + C_{12} X + C_{13} X^2 = (X - X_2)(X - X_3)$$
$$C_{21} + C_{22} X + C_{23} X^2 = (X - X_1)(X - X_3) \qquad \text{(A-19)}$$
$$C_{31} + C_{32} X + C_{33} X^2 = (X - X_1)(X - X_2)$$

Computing the coefficients of similar powers of X, the following relations are obtained for the C_{ij}:

$$C_{11} = X_2X_3 \qquad C_{12} = -(X_2 + X_3) \qquad C_{13} = 1$$
$$C_{21} = X_1X_3 \qquad C_{22} = -(X_1 + X_3) \qquad C_{23} = 1 \qquad \text{(A-20)}$$
$$C_{31} = X_1X_2 \qquad C_{32} = -(X_1 + X_2) \qquad C_{33} = 1$$

For the case considered here (N=3), since the grid points are equally spaced, one has

$$X_1 = 0 \qquad X_2 = 1/2 \qquad X_3 = 1 \qquad \text{(A-21)}$$

Substituting Eq. (A-21) into Eqs. (A-18) and (A-20), respectively, one obtains

$$\Pi_1(X_1) = 1/2 \qquad \Pi_2(X_2) = -1/4 \qquad \Pi_3(X_3) = 1/2 \qquad \text{(A-22)}$$

and

$$C_{11} = 1/2 \qquad C_{12} = -3/2 \qquad C_{13} = 1$$
$$C_{21} = 0 \qquad C_{22} = -1 \qquad C_{23} = 1 \qquad \text{(A-23)}$$
$$C_{31} = 0 \qquad C_{32} = -1/2 \qquad C_{33} = 1$$

Substituting Eqs. (A-22) and (A-23) into Eq. (A-15), one can obtain

$$[\,X\,]^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 4 & -4 \\ 0 & -1 & 2 \end{bmatrix} \tag{A-24}$$

Substitution of this into Eq. (A-12') results in

$$\begin{Bmatrix} w_{i1} \\ w_{i2} \\ w_{i3} \end{Bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 4 & -4 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 2X_i \end{Bmatrix} \tag{A-25}$$

Expanding this, one obtains

$$\begin{bmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ w_{13} & w_{23} & w_{33} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 4 & -4 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \tag{A-26}$$

After manipulation, one gets

$$\begin{bmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ w_{13} & w_{23} & w_{33} \end{bmatrix} = \begin{bmatrix} -3 & -1 & 1 \\ 4 & 0 & -4 \\ -1 & 1 & 3 \end{bmatrix} \tag{A-27}$$

Finally, taking the transpose of the above matrix, one can determine the weighting coefficients required as follows:

$$\begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} = \begin{bmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ w_{13} & w_{23} & w_{33} \end{bmatrix}^T = \begin{bmatrix} -3 & 4 & -1 \\ -1 & 0 & 1 \\ 1 & -4 & 3 \end{bmatrix} \qquad \text{(A-28)}$$

APPENDIX B

EXAMPLES OF WEIGHTING COEFFICIENTS

Typical weighting coefficients, A_{ij}, B_{ij}, C_{ij}, and D_{ij} for the first-, second-, third-, and fourth order-derivatives, respectively, are listed below for equally spaced sample points in the range $0 \leq X \leq 1$.

1. Three Nodal Points

$X = 0,\ 0.5,\ 1.0$

$$A_{ij} = \begin{bmatrix} -3 & 4 & -1 \\ -1 & 0 & 1 \\ 1 & -4 & 3 \end{bmatrix}$$

$$B_{ij} = \begin{bmatrix} 4 & -8 & 4 \\ 4 & -8 & 4 \\ 4 & -8 & 4 \end{bmatrix}$$

$$C_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. **Four Nodal Points**

$X = 0,\ 1/3,\ 2/3,\ 1$

$$A_{ij} = \frac{1}{2}\begin{bmatrix} -11 & 18 & -9 & 2 \\ -2 & -3 & 6 & -1 \\ 1 & -6 & 3 & 2 \\ -2 & 9 & -18 & 11 \end{bmatrix}$$

$$B_{ij} = \begin{bmatrix} 18 & -45 & 36 & -9 \\ 9 & -18 & 9 & 0 \\ 0 & 9 & -18 & 9 \\ -9 & 36 & -45 & 18 \end{bmatrix}$$

$$C_{ij} = \begin{bmatrix} -27 & 81 & -81 & 27 \\ -27 & 81 & -81 & 27 \\ -27 & 81 & -81 & 27 \\ -27 & 81 & -81 & 27 \end{bmatrix}$$

$$D_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that, for third-order weighting coefficients, three grid points are not sufficient and, for fourth-order weighting coefficients, four grid points are not sufficient. Thus, C_{ij} and D_{ij} are found to be zero in these respective case.

3. **Five Nodal Points**

X = 0, 0.25, 0.5, 0.75, 1.0

$$A_{ij} = \frac{1}{3}\begin{bmatrix} -25 & 48 & -36 & 16 & -3 \\ -3 & -10 & 18 & -6 & 1 \\ 1 & -8 & 0 & 8 & -1 \\ -1 & 6 & -18 & 10 & 3 \\ 3 & -16 & 36 & -48 & 25 \end{bmatrix}$$

$$B_{ij} = \frac{1}{3}\begin{bmatrix} 140 & -416 & 456 & -224 & 44 \\ 44 & -80 & 24 & 16 & -4 \\ -4 & 64 & -120 & 64 & -4 \\ -4 & 16 & 24 & -80 & 44 \\ 44 & -224 & 456 & -416 & 140 \end{bmatrix}$$

$$C_{ij} = \begin{bmatrix} -160 & 576 & -768 & 448 & -96 \\ -96 & 320 & -384 & 192 & -32 \\ -32 & 64 & 0 & -64 & 32 \\ 32 & -192 & 384 & -320 & 96 \\ 96 & -448 & 768 & -576 & 160 \end{bmatrix}$$

$$D_{ij} = \begin{bmatrix} 256 & -1024 & 1536 & -1024 & 256 \\ 256 & -1024 & 1536 & -1024 & 256 \\ 256 & -1024 & 1536 & -1024 & 256 \\ 256 & -1024 & 1536 & -1024 & 256 \\ 256 & -1024 & 1536 & -1024 & 256 \end{bmatrix}$$

APPENDIX C

ERROR ESTIMATION FOR A SIMPLY SUPPORTED BEAM

Consider the small deflection behavior of a simply supported beam under uniformly distributed load. Using the constitutive equation, one has

$$\frac{d^2y}{dx^2} = \frac{w}{2EI} (Lx - x^2) \qquad (C-1)$$

Normalizing x and y as

$$X \equiv \frac{x}{L} \qquad Y \equiv \frac{y}{\alpha} \qquad (C-2)$$

where α is the reference length, one obtains

$$\frac{d^2Y}{dX^2} = X - X^2 \qquad (C-3)$$

where

$$\alpha \equiv wL^4/2EI \qquad (C-4)$$

Applying differential quadrature yields

$$\sum_{j=1}^{N} B_{ij}\, Y_j = X_i - X_i^2 \qquad \text{(C-5)}$$

Setting N=5 and applying the boundary conditions

$$Y_1 = Y_5 = 0 \qquad \text{(C-6)}$$

one obtains

$$\sum_{j=2}^{4} B_{ij}\, Y_j = X_i - X_i^2 \qquad ; \quad i = 2,3,4 \qquad \text{(C-7)}$$

Thus, in matrix form

$$\begin{bmatrix} B_{22} & B_{23} & B_{24} \\ B_{32} & B_{33} & B_{34} \\ B_{42} & B_{43} & B_{44} \end{bmatrix} \begin{Bmatrix} Y_2 \\ Y_3 \\ Y_4 \end{Bmatrix} = \begin{Bmatrix} X_2 - X_2^2 \\ X_3 - X_3^2 \\ X_4 - X_4^2 \end{Bmatrix} \qquad \text{(C-8)}$$

Here, the B_{ij}'s, the X_i's are known; thus, the Y_i's can be determined.

In order to investigate the error, the above matrix is solved analytically and the results are compared to the exact ones in Table C-1. There is no difference between the exact and the analytical solutions for this problem with N=5. Based on the analysis in Section 2.4, one can determine the error. Using Eqs. (C-5) and (C-9), and the exact solution of this problem,

$$Y = \frac{1}{12} (2X^3 - X^4 - X) \qquad (C-9)$$

one gets

$$Y^{(N)}(x) = 0 \quad ; \quad K = 0 \quad \text{with } N=5 \qquad (C-10)$$

Thus, the DQ results are expected to be coincident with the exact results. For comparison, Tables C-2 and C-3 list the results for N=3 and N=4. In Table C-2, the maximum possible error calculated from the previous error estimation is $R''(x) \leq 0.5$, and in Table C-3, it is 0.1875.

It is expected that in higher order problems, even with a large number of grid points, an error can be found due to the characteristics of applying the boundary conditions at the ends.

Table C-1. **Nondimensionalized Deflections for a Simply Supported Beam Under Uniformly Distributed Load with N=5**

X	Exact Solution	Analytical Results by DQM*
0	0	0
0.25	19/1024	19/1024
0.50	5/192	5/192
0.75	19/1024	19/1024
1	0	0

Table C-2. **Nondimensionalized Deflections for a Simply Supported Beam Under Uniformly Distributed Load with N=3**

X	Exact Solution	Analytical Results by DQM*
0	0	0
0.50	5/192	6/192
1	0	0

Table C-3. **Nondimensionalized Deflections for a Simply Supported Beam Under Uniformly Distributed Load with N=4**

X	Exact Solution	Analytical Results by DQM*
0	0	0
1/3	0.0226337	0.0246914
2/3	0.0226337	0.0246914
1	0	0

* The solution was carried out analytically to avoid any additional errors due to round off.

APPENDIX D

DETERMINATION OF OPTIMAL VALUE FOR N

Consider a differential equation

$$\frac{d^2Y_e}{dX^2} = g(X) \qquad \text{(D-1)}$$

In differential quadrature formulation,

$$\frac{d^2Y_q}{dX^2} = g(X) + R''(X) \qquad \text{(D-2)}$$

where $R''(X)$ represents the error of the DQ approximation. Subtracting (D-2) from (D-1) yields

$$\frac{d^2Y_e}{dX^2} - \frac{d^2Y_q}{dX^2} = - R''(X) \qquad \text{(D-3)}$$

Using the expression for $R''(X)$ from section 2.4, one obtains a minimum error if

$$R''(X) = Y^{(N)}(\bar{x}) \frac{h^{N-2}}{(N-2)!} = 0 \qquad \text{(D-4)}$$

Thus, the N which satisfies Eq. (D-4) will give the exact solution. It is thought that this could be achieved by either finding N which makes $Y^{(N)}(x)=0$ or letting h^{N-2} be close to zero since h is always less than 1. For example, consider the same problem as in Appendix C. Using the above approach, one can find that N=5 results in $Y^{(N)}(x)$ being zero. Thus, five grid points are enough to give the exact solution.