تمرین ۲

تاریخ تحویل ۸۵/۹/۲۰

32. By hand calculation, find the natural cubic spline interpolant for this table:

x	1	2	3	4	5
y	0	1	0	1	0

33. Find a cubic spline over knots $-1, 0$, and 1 such that the following conditions are satisfied: $S''(-1) = S''(1) = 0$, $S(-1) = S(1) = 0$, and $S(0) = 1$.

34. This problem and the next two lead to a more efficient algorithm for natural cubic spline interpolation in the case of equally spaced knots. Let $h_i = h$ in Equation (5), and replace the parameters z_i by $q_i = h^2 z_i/6$. Show that the new form of Equation (5) is then

$$S_i(x) = q_{i+1}\left(\frac{x - t_i}{h}\right)^3 + q_i\left(\frac{t_{i+1} - x}{h}\right)^3 + (y_{i+1} - q_{i+1})\left(\frac{x - t_i}{h}\right) + (y_i - q_i)\left(\frac{t_{i+1} - x}{h}\right)$$

35. (Continuation) Establish the new continuity conditions:

$$q_{i-1} + 4q_i + q_{i+1} = y_{i+1} - 2y_i + y_{i-1} \qquad (1 \leqq i \leqq n-1)$$
$$q_0 = q_n = 0$$

36. (Continuation) Show that the parameters q_i can be determined by backward recursion as follows:

$$q_n = 0 \qquad q_{n-1} = \beta_n \qquad q_i = \alpha_i q_{i+1} + \beta_i \qquad (i = n-2, n-3, \ldots, 1)$$

where the coefficients α_i and β_i are generated by ascending recursion from the formulas

$$\begin{array}{ll} \alpha_1 = 0 & \alpha_i = -(\alpha_{i-1} + 4)^{-1} \qquad (i = 1, 2, \ldots, n) \\ \beta_1 = 0 & \beta_i = -\alpha_i(y_{i+1} - 2y_i + y_{i-1} - \beta_{i-1}) \qquad (i = 1, 2, \ldots, n) \end{array}$$

(This algorithm, which is stable and efficient, is due to MacLeod [1973].)

37. Prove that if $S(x)$ is a spline of degree k on $[a, b]$, then $S'(x)$ is a spline of degree $k - 1$.

38. How many coefficients are needed to define a piecewise quartic (fourth-degree) function with $n + 1$ knots? How many conditions will be imposed if the piecewise quartic function is to be a quartic spline? Justify your answers.

39. Determine whether this function is a natural cubic spline:

$$S(x) = \begin{cases} x^3 + 3x^2 + 7x - 5 & -1 \leqq x \leqq 0 \\ -x^3 + 3x^2 + 7x - 5 & 0 \leqq x \leqq 1 \end{cases}$$

5. Draw a free-form curve on graph paper, making certain that the curve is the graph of a function. Then read values of your function at a reasonable number of points, say 10 to 50, and compute the cubic spline function that takes those values. Compare the freely drawn curve to the graph of the cubic spline.

6. Draw a spiral (or other curve that is not a function) and reproduce it by spline functions as follows: Select points on the curve and label them $t = 0, 1, \ldots, n$. For each value of t, read off the x- and y-coordinates of the point, thus producing a table:

t	0	1	$\cdots$	n
x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

Then fit $x = S(t)$ and $y = \overline{S}(t)$, where S and $\overline{S}$ are natural cubic spline interpolants. S and $\overline{S}$ give a parametric representation of the curve (see the figure).

$$S_i(x) = \frac{z_{i+1}}{6h_i}(x - t_i)^3 + \frac{z_i}{6h_i}(t_{i+1} - x)^3 + \left(\frac{y_{i+1}}{h_i} - \frac{h_i}{6} z_{i+1}\right)(x - t_i) + \left(\frac{y_i}{h_i} - \frac{h_i}{6} z_i\right)(t_{i+1} - x)$$

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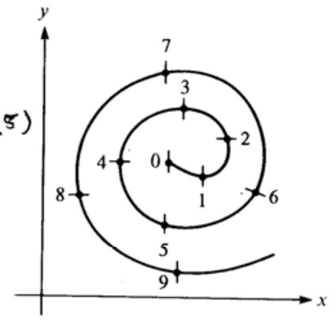

8. Write a program to estimate $\int_a^b f(x)\,dx$, assuming that we know the values of f at only certain prescribed knots $a = t_0 < t_1 < \cdots < t_n = b$. Approximate f first by an interpolating cubic spline and then compute the integral of it using Equation (5).

9. Write a procedure to estimate $f'(x)$ for any x in $[a, b]$, assuming that we know only the values of f at knots $a = t_0 < t_1 < \cdots < t_n = b$.