Chapter 6

Finite Elements for Plane Solids

6.1 INTRODUCTION

All real-life structures are three-dimensional. It is engineers who make the approximation as a one-dimensional (e.g., beam) or a two-dimensional (e.g., plate or plane solid) structure. In Chapter 1 we explained in detail the conditions under which such approximation could be made. When the stresses on a plane normal to one of the axes are approximately zero, then we say that the solid is in the state of plane stress. Similarly, when the corresponding strains are zero, the solid is in the state of plane strain. A two-dimensional solid is also called a plane solid. Some examples of plane solids are (1) a thin plate subjected to in-plane forces and (2) a very thick solid with constant cross-section in the thickness direction. In this chapter, we will discuss when an engineering problem can be assumed to be two-dimensional and how to solve such a problem using two-dimensional finite elements. We will introduce three different types of plane elements. Every element has its own characteristics. In order to use the finite element appropriately, thorough understandings of capabilities and limitations of each element are required.

In general, two-dimensional elasticity problems can be expressed by a system of coupled second-order partial differential equations. Based on the constraints imposed in the thickness direction, a two-dimensional problem can be considered as either a plane stress or plane strain problem. Although the two problems are different, the equations developed for plane stress problems can be used for plane strain case by modifying the elastic constants.

In the plane solid problem, the main variables are the displacements in the coordinate directions. After solving for the displacements, stresses and strains can be calculated from the derivatives of displacements. The displacements are calculated using the fact that the structure is in equilibrium when the total potential energy has its minimum value. This will yield a matrix equation similar to the beam problem in Chapter 4.

6.2 TYPES OF TWO-DIMENSIONAL PROBLEMS

6.2.1 Governing Differential Equations

In two-dimensional problems, the stresses and strains are independent of the coordinate in the thickness direction (usually the z-axis). By setting all the derivatives with respect to z-coordinate in Eq. (1.68) to zero, we obtain the governing differential equations for plane problems as

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases} \tag{6.1}$$

where b_x and b_y are the body forces.

Let u and v be the displacement in the x- and y-directions, respectively. From Eqs. (1.39) through (1.45), the strain components in a plane solid are defined as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{6.2}$$

In addition, the stresses and strains are related by the following constitutive relation:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \Leftrightarrow \{\sigma\} = [C]\{\varepsilon\} \tag{6.3}$$

Substituting the stress-strain relations in Eq. (6.3) and strain-displacement relations in Eq. (6.2) in the equilibrium equations in Eq. (6.1), we obtain a pair of second-order partial differential equations in two variables, $u(x,y)$ and $v(x,y)$. The explicit form of the equations is available in textbooks on elasticity, e.g., Timoshenko and Goodier.[1]

The differential equation must be accompanied by boundary conditions. Two types of boundary conditions can be defined. The first one is the boundary in which the values of displacements are prescribed (*essential boundary condition*). The other is the boundary in which the tractions are prescribed (*natural boundary condition*). The boundary conditions can be formally stated as

$$\begin{aligned} \mathbf{u} &= \mathbf{g}, \quad \text{on } S_g \\ \boldsymbol{\sigma}\mathbf{n} &= \mathbf{T}, \quad \text{on } S_T \end{aligned} \tag{6.4}$$

where S_g and S_T, respectively, are the boundaries where the displacement and traction boundary conditions are prescribed. The objective is to determine the displacement field $u(x,y)$ and $v(x,y)$ that satisfies the differential equation (6.1) and the boundary conditions in Eq. (6.4). Now we will discuss the stress-strain relations in Eq. (6.3) for the two different plane problems.

6.2.2 Plane Stress Problems

Plane stress conditions exist when the thickness dimension (usually the z-direction) is much smaller than the length and width dimensions of a solid. Since stress at the two surfaces normal to the z-axis are zero, it is assumed that stresses in the normal direction are zero throughout the body; i.e., $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$. In such a case, the structure can be modeled in two dimensions. An example of the plane stress problem is a thin plate or disk with applied in-plane forces (see Figure 6.1). If an out-of-plane force (e.g., transverse pressure in the z-direction) is applied, then the problem can be assumed plane stress only when the applied pressure load is much smaller than the in-plane stresses such as σ_{xx}.

- Non-zero stress components: σ_{xx}, σ_{yy}, τ_{xy}.
- Non-zero strain components: ε_{xx}, ε_{yy}, γ_{xy}, ε_{zz}.

For linear isotropic materials, the stress-strain relation can be written as (see Section 1.3)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{6.5}$$

$$\Leftrightarrow \{\sigma\} = [\mathbf{C}_D]\{\varepsilon\}$$

[1] S.P. Timoshenko and J.N. Goodier, Theory of Elasticity, McGraw-Hill, NY, 1984.

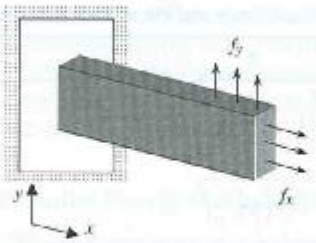


Figure 6.1 Thin plate with in-plane applied forces

where $[\mathbf{C}_\sigma]$ is the stress-strain matrix or elasticity matrix for the plane stress problem. It should be noted that the normal strain ε_{zz} in the thickness direction is not zero; it can be calculated from the following relation:

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \tag{6.6}$$

6.2.3 Plane Strain Problems

A state of plane strain will exist in a solid when the thickness dimension is much larger than other two dimensions. When the deformation in the thickness direction is constrained, the solid is assumed to be in a state of plane strain even if the thickness dimension is small. A proper assumption is that that strains with z subscript are zero i.e., $\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$. In such a case, it is sufficient to model a slice of the solid with unit thickness. Some examples of plane strain problems are the retaining wall of a dam and long cylinder such as a gun barrel (see Figure 6.2).

- Non-zero stress components: σ_{xx}, σ_{yy}, τ_{xy}, σ_{zz}.
- Non-zero strain components: ε_{xx}, ε_{yy}, γ_{xy}.

For linear isotropic materials, the stress-strain relations under plane strain conditions can be written as (see Section 1.3)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{6.7}$$
$$\Leftrightarrow \{\sigma\} = [\mathbf{C}_\varepsilon]\{\varepsilon\}$$

where $[\mathbf{C}_\varepsilon]$ is the stress-strain matrix for plane strain problem. It should be noted that the transverse stress σ_{zz} is not equal to zero in plane strain case, and it can be calculated from the following relation:

$$\sigma_{zz} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\varepsilon_{xx} + \varepsilon_{yy}) \tag{6.8}$$

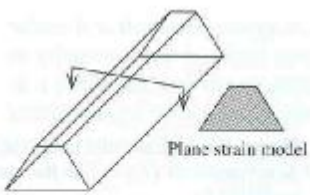


Figure 6.2 Dam structure with plane strain assumption

Table 6.1 Material Property Conversion Between Plane Strain and Plane Stress Problems

From → To	E	ν
Plane strain → Plane stress	$E\left[1-\left(\frac{\nu}{1+\nu}\right)^2\right]$	$\frac{\nu}{1+\nu}$
Plane stress → Plane strain	$\frac{E}{1-\left(\frac{\nu}{1-\nu}\right)^2}$	$\frac{\nu}{1-\nu}$

6.2.4 Equivalence between Plane Stress and Plane Strain Problems

Although plane stress and plane strain problems are different by definition, they are quite similar from the computational viewpoint. Thus, it is possible to use the plane strain formulation and solve the plane stress problem. In such case, two material properties, E and ν, need to be modified. Similarly, it is also possible to convert the plane stress formulation into the plane strain formulation. Table 6.1 summarizes the conversion relations.

6.3 PRINCIPLE OF MINIMUM POTENTIAL ENERGY

Similar to the beam-bending problem in Chapter 4, the principle of minimum potential energy can be used to derive the finite element equations for the two-dimensional plane solid problems.

6.3.1 Strain Energy in a Plane Solid

Consider a plane elastic solid as illustrated in Figure 6.3. The *strain energy* is a form of energy that is stored in the solid due to the elastic deformation. Formally, it can be defined as

$$\begin{aligned} U &= \frac{1}{2}\iiint_{\text{volume}} \{\varepsilon\}^T\{\sigma\}dV \\ &= \frac{h}{2}\iint_{\text{area}} \{\varepsilon\}^T\{\sigma\}dA \\ &= \frac{h}{2}\iint_{\text{area}} \{\varepsilon\}^T[\mathbf{C}]\{\varepsilon\}dA \end{aligned} \tag{6.9}$$

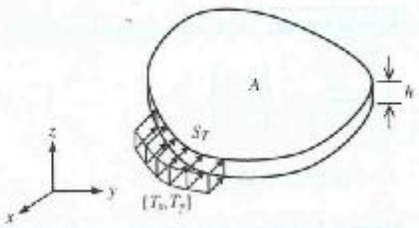


Figure 6.3 A plane solid under the distributed load $\{T_x, T_y\}$ on the traction boundary S_T

where h is the thickness of the plane solid ($h = 1$ for plane strain) and $[\mathbf{C}] = [\mathbf{C}_\sigma]$ for plane stress and $[\mathbf{C}] = [\mathbf{C}_\varepsilon]$ for plane strain. Since stress and strain are constant throughout the thickness, the volume integral is changed to area integral by multiplying by the thickness in the second line in Eq. (6.9). The linear elastic relation in Eq. (6.3) has been used in the last line of Eq. (6.9).

6.3.2 Potential Energy of Applied Loads

When a force acting on a body moves through a small distance, it loses its potential to do additional work, and hence its potential energy is given by the negative of product of the force and corresponding displacement. For example, when concentrated forces are applied to the solid, the potential energy becomes

$$V = -\sum_{i=1}^{ND} F_i q_i \tag{6.10}$$

where F_i is the i-th force, q_i is the displacement in the direction of the force, and ND is the total number of concentrated forces acting on the body. The negative sign indicates that the potential energy decreases as the force has expended some energy performing the work given by the product of force and corresponding displacement.

When distributed forces, such as a pressure load, act on the edge of a body (see Figure 6.3), the summation sign in the above expression is replaced by integration over the edge of the body as shown below:

$$\begin{aligned} V &= -h\int_{S_T} (T_x u + T_y v)dS \\ &= -h\int_{S_T} \begin{bmatrix} u & v \end{bmatrix} \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} dS \\ &= -h\int_{S_T} \{\mathbf{u}\}^T \{\mathbf{T}\} dS \end{aligned} \tag{6.11}$$

where T_x and T_y are the components of applied surface forces in the x- and y-direction, respectively.

If body forces (forces distributed over the volume) are present, work done by these forces can be computed in a similar manner. The gravitational force is an example of body force. In this case, integration should be performed over the volume. We will discuss this further, when we derive the finite element equations.

6.3.3 Total Potential Energy

As with the beam problem, the *potential energy* is defined as the sum of the strain energy and the potential energy of applied loads:

$$\Pi = U + V \tag{6.12}$$

where U is the strain energy and V is the potential energy of applied loads. The principle of minimum total potential energy states that of all possible displacement configurations of a solid/structure, the equilibrium configuration corresponds to the minimum total potential energy. That is, at equilibrium, we have

$$\frac{\partial \Pi}{\partial \{\mathbf{u}\}} = 0 \Rightarrow \frac{\partial \Pi}{\partial u_1} = 0, \frac{\partial \Pi}{\partial u_2} = 0 \cdots \frac{\partial \Pi}{\partial u_N} = 0 \tag{6.13}$$

where $u_1, u_2, \ldots, u_N$ are the displacements that define the deformed configuration of the body. In finite element analysis, the deformation of the body is defined in terms of the displacements of the nodes. In the following sections, we will use the principal of minimum total potential energy to derive finite element equations for different types of elements.

6.4 CONSTANT STRAIN TRIANGULAR (CST) ELEMENT

In finite element analysis, a plane solid can be divided into a number of contiguous elements. A simplest way of dividing a plane solid is to use triangular elements. Figure 6.4 shows a plane solid that is divided by triangular elements. Each element shares its edge and two corner nodes with an adjacent element, except for those on the boundary. The three vertices of a triangle are the nodes of that element as shown in Figure 6.4. The first node of an element can arbitrarily be chosen. However, the sequence of the nodes 1, 2, and 3 should be in the counter-clockwise direction. Each node has two displacements, u and v, respectively, in the x- and y-directions.

The displacements within the element are interpolated in terms of the nodal displacements using shape functions. In the polynomial approximation, the displacement has to be a linear function in x and y because displacement information is available only at three points (nodes), and a linear polynomial has three unknown coefficients. Since displacement is a linear function, strain and stress are constant within an element, and that is why a triangular element is called a *constant strain triangle element*.

6.4.1 Displacement and Strain Interpolation

The first step in deriving the finite element matrix equation is to interpolate the displacement function in terms of the nodal displacements. Let the x- and y-directional displacements be $u(x,y)$ and $v(x,y)$, respectively. Since the two-coordinates are perpendicular to each other (orthogonal), the $u(x,y)$ and $v(x,y)$ are independent of each other. Hence, $u(x,y)$ needs to be interpolated in terms of u_1, u_2, and u_3, and $v(x,y)$ in terms of v_1, v_2, and v_3. It is obvious that the interpolation function must be a three-term polynomial in x and y. Since we must have rigid body displacements (constant displacements) and constant strain terms in the interpolation function, the displacement interpolation must be of the following form:

$$\begin{cases} u(x, y) = \alpha_1 - \alpha_2 x + \alpha_3 y \\ v(x, y) = \beta_1 + \beta_2 x + \beta_3 y \end{cases} \tag{6.14}$$

where α's and β's are constants. In finite element analysis, we would like to replace the constants by the nodal displacements. Let us consider x-directional displacements, which

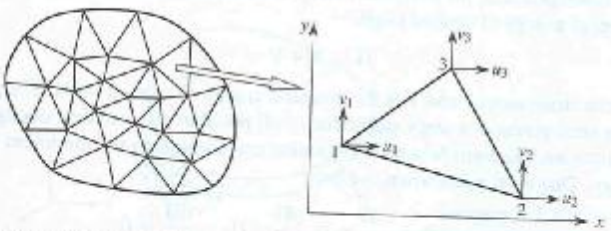


Figure 6.4 Constant strain triangular (CST) element

are u_1, u_2, and u_3. At Node 1, for example, x and y take the values of x_1 and y_1, respectively, and the nodal displacement is u_1. If we repeat this for the other two nodes, we obtain the following three simultaneous equations:

$$\begin{cases} u(x_1, y_1) \equiv u_1 = \alpha_1 - \alpha_2 x_1 + \alpha_3 y_1 \\ u(x_2, y_2) \equiv u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 \\ u(x_3, y_3) \equiv u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 \end{cases} \tag{6.15}$$

In matrix notation, the above equations can be written as

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{6.16}$$

If the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are not on a straight line, then the inverse of the above coefficient matrix exists. Thus, we can calculate the unknown coefficients as

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{6.17}$$

where A is the area of the triangle and

$$\begin{cases} f_1 = x_2 y_3 - x_3 y_2, & b_1 = y_2 - y_3, & c_1 = x_3 - x_2 \\ f_2 = x_3 y_1 - x_1 y_3, & b_2 = y_3 - y_1, & c_2 = x_1 - x_3 \\ f_3 = x_1 y_2 - x_2 y_1, & b_3 = y_1 - y_2, & c_3 = x_2 - x_1 \end{cases} \tag{6.18}$$

The area A of the triangle can be calculated from

$$A = \frac{1}{2} \det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \tag{6.19}$$

Note that the determinant in Eq. (6.19) is zero when three nodes are collinear. In such a case, the area of the triangular element is zero and we cannot uniquely determine the three coefficients.

A similar procedure can be applied for y-directional displacement $v(x, y)$, and the unknown coefficients β_i, $(i = 1, 2, 3)$ are determined using the following equation:

$$\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \tag{6.20}$$

After calculating α_i and β_i, the displacement interpolation can be written as

$$u(x, y) = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \text{ and } v(x, y) = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \tag{6.21}$$

where the shape functions are defined by

$$\begin{cases} N_1(x, y) = \dfrac{1}{2A}(f_1 + b_1 x + c_1 y) \\ N_2(x, y) = \dfrac{1}{2A}(f_2 + b_2 x + c_2 y) \\ N_3(x, y) = \dfrac{1}{2A}(f_3 + b_3 x + c_3 y) \end{cases} \tag{6.22}$$

Note that N_1, N_2, and N_3 are linear functions of x- and y-coordinates. Thus, interpolated displacement varies linearly in each coordinate direction.

To make the derivations simple, we would rewrite the interpolation relation in Eq. (6.21) in matrix form. Let $\{\mathbf{u}\} = \{u, v\}^T$ be the displacement vector at any point (x, y). The interpolation can be written in the matrix notation by

$$\{\mathbf{u}\} \equiv \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

or

$$\boxed{\{\mathbf{u}(x, y)\} = [\mathbf{N}(x, y)]\{\mathbf{q}\}} \tag{6.23}$$

Equation (6.23) is the critical relation in finite element approximation. When a point (x, y) within a triangular element is given, the shape function [**N**] is calculated at this point. Then the displacement at this point can be calculated by multiplying this shape function matrix with the nodal displacement vector $\{\mathbf{q}\}$. Thus, if we solve for the nodal displacements, we can calculate the displacement everywhere in the element. Note that the nodal displacements will be evaluated using the principle of minimum total potential energy in the following section.

After calculating displacement within an element, it is possible to calculate the strain by differentiating the displacement with respect to x and y. From the expression in Eq. (6.23), it can be noted that the nodal displacements are constant, but the shape functions are functions of x and y. Thus, the strain can be calculated by differentiating the shape function with respect to the coordinates. For example, ε_{xx} can be written as

$$\varepsilon_{xx} \equiv \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} N_i(x, y)u_i\right) = \sum_{i=1}^{3} \frac{\partial N_i}{\partial x} u_i = \sum_{i=1}^{3} \frac{b_i}{2A} u_i \tag{6.24}$$

Note that u_1, u_2, and u_3 are nodal displacements and they are independent of coordinate x. Thus, only the shape function is differentiated with respect to x. Similar calculation can be carried out for ε_{yy} and γ_{xy}. Using the matrix notation, we have

$$\{\varepsilon\} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \equiv [\mathbf{B}]\{\mathbf{q}\} \tag{6.25}$$

It may be noted that the [**B**] matrix is constant and depends only on the coordinates of the three nodes of the triangular element. Thus, one can anticipate that if this element is used, then the strains will be constant over a given element and will depend only on nodal displacements. Hence, this element is called the **Constant Strain Triangular Element** or **CST** element.

The interpolation of displacement in Eq. (6.23) and the interpolation of strain in Eq. (6.25) are used for approximating the strain energy and potential energy of applied loads.

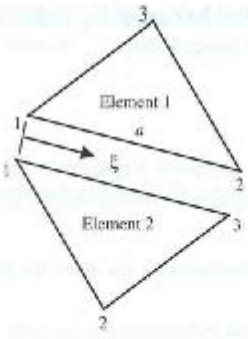


Figure 6.5 Two adjacent constant strain triangular (CST) elements

6.4.2 Properties of the CST Element

Before we derive the strain energy, it may be useful to study some interesting aspects of the CST element. Since the displacement field is assumed to be a linear function in x and y, one can show that the triangular element deforms into another triangle when forces are applied. Furthermore, an imaginary straight line drawn within an element before deformation becomes another straight line after deformation.

Let us consider the displacements of points along one of the edges of the triangle. Consider the points along the edge 1-2 of Element 1 in Figure 6.5. These points can be conveniently represented by a coordinate ξ. The coordinate $\xi = 0$ at Node 1 and $\xi = a$ at Node 2. Along this edge, x and y are related to ξ. By substituting this relation in the displacement functions, one can express the displacements of points on the edge 1-2 as a function of ξ. It can be easily shown that the displacement functions, for both u and v, must be linear in ξ, i.e.,

$$\begin{cases} u(\xi) = \gamma_1 - \gamma_2 \xi \\ v(\xi) = \gamma_3 - \gamma_4 \xi \end{cases}$$

where γ's are constants to be determined. Since the variation of displacement is linear, it might be argued that the displacements should depend only on u_1 and u_2, and not on u_3. Then, the displacement field along the edge 1-2 takes the following form:

$$\begin{cases} u(\xi) = \left(1 - \dfrac{\xi}{a}\right) u_1 + \dfrac{\xi}{a} u_2 = H_1(\xi) u_1 + H_2(\xi) u_2 \\ v(\xi) = \left(1 - \dfrac{\xi}{a}\right) v_1 + \dfrac{\xi}{a} v_2 = H_1(\xi) v_1 + H_2(\xi) v_2 \end{cases} \tag{6.26}$$

where H_1 and H_2 are shape functions defined along the edge 1-2 and a is the length of edge 1-2. One can also note that a condition called *inter-element displacement compatibility* is satisfied by triangular elements. This condition can be described as follows. After the loads are applied and the solid is deformed, the displacements at any point in an element can be computed from the nodal displacements of that particular element and the interpolation functions in Eq. (6.23). Consider a point on a common edge of two adjacent elements. This point can be considered as belonging to either of the elements. Then the nodes of either triangle can be used in interpolating the displacements of this point. However, one must obtain a unique set of displacements independent of the choice of the element. This can be true only if the displacements of the points depend only on the nodes

common to both elements. In fact, this will be satisfied because of Eq. (6.26). Thus, the CST element satisfies the inter-element displacement compatibility.

EXAMPLE 6.1 ***Interpolation in a Triangular Element***

Consider two triangular elements shown in Figure 6.6. The nodal displacements are given as $\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{-0.1, 0, 0.1, 0, -0.1, 0, 0.1, 0\}$. Calculate displacements and strains in both elements.

SOLUTION Element 1 has nodes 1-2-4. Then, using the nodal coordinates, we can derive the shape functions as shown below:

$$\begin{array}{lll} x_1 = 0 & x_2 = 1 & x_3 = 0 \\ y_1 = 0 & y_2 = 0 & y_3 = 1 \\ f_1 = 1 & f_2 = 0 & f_3 = 0 \\ b_1 = -1 & b_2 = 1 & b_3 = 0 \\ c_1 = -1 & c_2 = 0 & c_3 = 1 \end{array}$$

In addition, the area of the element is 0.5. Thus, from Eq. (6.22) the shape functions can be derived as

$$\begin{aligned} N_1(x, y) &= 1 - x - y \\ N_2(x, y) &= x \\ N_3(x, y) &= y \end{aligned}$$

Then, the displacements in Element 1 can be interpolated as

$$u^{(1)}(x, y) = \sum_{i=1}^{3} N_i(x, y) u_i = 0.1(2x - 1)$$

$$v^{(1)}(x, y) = \sum_{i=1}^{3} N_i(x, y) v_i = 0.0$$

Strains can be calculated from Eq. (6.25), or directly differentiating the above expressions for displacements, as

$$\varepsilon_{xx}^{(1)} = \frac{\partial u^{(1)}}{\partial x} = 0.2$$

$$\varepsilon_{yy}^{(1)} = \frac{\partial v^{(1)}}{\partial y} = 0.0$$

$$\gamma_{xy}^{(1)} = \frac{\partial u^{(1)}}{\partial y} + \frac{\partial v^{(1)}}{\partial x} = 0.0$$

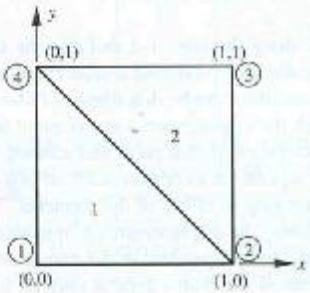


Figure 6.6 Interpolation of displacements in triangular elements

Element 2 connects Nodes 2-3-4. Thus, using the nodal coordinates, we can build the shape functions, as

$$\begin{array}{lll} x_1 = 1 & x_2 = 1 & x_3 = 0 \\ y_1 = 0 & y_2 = 1 & y_3 = 1 \\ f_1 = 1 & f_2 = -1 & f_3 = 1 \\ b_1 = 0 & b_2 = 1 & b_3 = -1 \\ c_1 = -1 & c_2 = 1 & c_3 = 0 \end{array}$$

In addition, the area of the element is 0.5. Thus, from Eq. (6.22), the shape functions can be obtained as

$$\begin{aligned} N_1(x, y) &= 1 - y \\ N_2(x, y) &= x + y - 1 \\ N_3(x, y) &= 1 - x \end{aligned}$$

Then, the displacements of Element 2 can be interpolated as

$$u^{(2)}(x, y) = \sum_{i=1}^{3} N_i(x, y) u_i = 0.1(3 - 2x - 2y)$$
$$v^{(2)}(x, y) = \sum_{i=1}^{3} N_i(x, y) v_i = 0.0$$

Strains can be calculated from Eq. (6.25), or directly differentiating the above expressions for displacements, as

$$\varepsilon_{xx}^{(2)} = \frac{\partial u^{(2)}}{\partial x} = -0.2$$
$$\varepsilon_{yy}^{(2)} = \frac{\partial v^{(2)}}{\partial y} = 0.0$$
$$\gamma_{xy}^{(2)} = \frac{\partial u^{(2)}}{\partial y} + \frac{\partial v^{(2)}}{\partial x} = -0.2$$

Note that the displacements are linear and the strains are constant in each element. From the given nodal displacements, it is clear that the top edge has strain $\varepsilon_{xx} = -0.2$, while the bottom edge has $\varepsilon_{xx} = 0.2$. The strain varies linearly along the y-coordinate. However, the triangular element cannot represent this change and provides constant values of $\varepsilon_{xx} = 0.2$ for Element 1 and $\varepsilon_{xx} = -0.2$ for Element 2. In general, if a plane solid is under the constant strain states, the CST element will provide accurate solutions. However, if the strain varies in the solid, then the CST element cannot represent it accurately. In such a case, many elements should be used to approximate it as a series of step functions. Note that the strains along the interface between two elements are discontinuous.

6.4.3 Strain Energy

Let us calculate the strain energy in a typical triangular element, say Element e. In Eq. (6.9), the strain energy of the plane solid was derived in terms of strains and the elasticity matrix [C]. Substituting for strains from Eq. (6.25), we obtain

$$\begin{aligned} U^{(e)} &= \frac{h}{2} \iint_A \{\varepsilon\}^T [\mathbf{C}] \{\varepsilon\} dA^{(e)} \\ &= \frac{h}{2} \{\mathbf{q}^{(e)}\}^T \iint_A \underset{6\times3}{[\mathbf{B}]^T} \underset{3\times3}{[\mathbf{C}]} \underset{3\times6}{[\mathbf{B}]} dA \{\mathbf{q}^{(e)}\} \\ &= \frac{1}{2} \{\mathbf{q}^{(e)}\}^T \underset{6\times6}{[\mathbf{k}^{(e)}]} \{\mathbf{q}^{(e)}\} \end{aligned} \tag{6.27}$$

where $[\mathbf{k}^{(e)}]$ is the *element stiffness matrix* of the triangular element. The column vector $\mathbf{q}^{(e)}$ is the displacement of the three nodes that belong to the element. The dimension of $[\mathbf{k}^{(e)}]$ is 6×6. In the case of the triangular element, all entries in matrices [B] and [C] are constant and can be integrated easily. After integration, the element stiffness matrix takes the form

$$\boxed{[\mathbf{k}^{(e)}] = hA[\mathbf{B}]^T[\mathbf{C}][\mathbf{B}]} \tag{6.28}$$

where A is the area of the plane element. Using the expression for [B] in Eq. (6.25) and stress-strain relation in Eq. (6.5), the element stiffness matrix can be calculated.

One may note that in the case of truss and frame elements, we used a transformation matrix [T] in deriving the element stiffness matrix. However, in the present case, we have used the global coordinate system in the derivation of $[\mathbf{k}^{(e)}]$, and that is the reason for not using a transformation matrix. In some cases, however, it is required to define the element in a local coordinate system. For example, if the material is not isotropic, it will have a specific directional property. In such a case, $[\mathbf{k}^{(e)}]$ is first derived in a local coordinate system and transformed to the global coordinates by multiplying by appropriate transformation matrices.

The strain energy of the entire solid is simply the sum of the element strain energies. That is,

$$U = \sum_{e=1}^{NE} U^{(e)} = \frac{1}{2}\sum_{e=1}^{NE} \{\mathbf{q}^{(e)}\}^T[\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\} \tag{6.29}$$

where NE is the number of elements in the model. The superscript (e) in $\mathbf{q}^{(e)}$ implies that it is the vector of displacements or degrees-of-freedom (DOFs) of Element e. The summation in the above equation leads to the assembling of the element stiffness matrices into the structural stiffness matrix.

$$U = \frac{1}{2}\{\mathbf{Q}_s\}^T[\mathbf{K}_s]\{\mathbf{Q}_s\} \tag{6.30}$$

where $\{\mathbf{Q}_s\}$ is the column vector of all displacements in the model and $[\mathbf{K}_s]$ is the structural stiffness matrix obtained by assembling the element stiffness matrices.

6.4.4 Potential Energy of Concentrated Forces at Nodes

The next step is to calculate the potential energy of external forces. We will consider three different types of applied forces. The first type is concentrated forces at nodes. It may be noted that the element expects two forces, one in the x-direction and the other in the y-direction, at each node. In general, the potential energy of concentrated nodal forces can be written as

$$V = -\sum_{i=1}^{ND}(F_{ix}u_i + F_{iy}v_i) \equiv -\{\mathbf{Q}_s\}^T\{\mathbf{F}_N\} \tag{6.31}$$

where $\{\mathbf{F}_N\} = [F_{1x} \quad F_{1y} \quad \cdots \quad F_{NDx} \quad F_{NDy}]^T$ is the vector of applied nodal forces and ND is the number of nodes in the solid. The contribution of a particular node to the potential energy will be zero if no force is applied at the node, or the displacement of the node becomes zero. The above potential energy of concentrated forces does not include supporting reaction because the displacement at those nodes will be zero.

6.4.5 Potential Energy of Distributed Forces along Element Edges

The second type of applied force is the distributed force (traction) on the side surface of the plane solid. In the plane solid, the traction is assumed to be a constant through the thickness. Let the surface traction force $\{\mathbf{T}\} = \{T_x, T_y\}^T$ is applied on the element edge 1-2 as shown in Figure 6.7. The unit of the surface traction is Pa (N/m^2) or psi. Since the force is distributed along the edge, the potential energy of the surface traction force must defined in the form of integral as

$$V^{(e)} = -h\int_{S_T}\{\mathbf{u}(s)\}^T\{\mathbf{T}(s)\}ds = -\{\mathbf{d}\}^T h\int_{S_T}[\mathbf{H}(s)]^T\{\mathbf{T}(s)\}ds \tag{6.32}$$

where $\{u\}^T = [u(s) \quad v(s)]$ is the vector of displacements along the Edge 1-2, $\{T\}^T = [T_x(s) \quad T_y(s)]$ is the vector of applied tractions along the Edge 1-2, $\{d\}^T = [u_1 \quad v_1 \quad u_2 \quad v_2]$ is the vector of displacements of Nodes 1 and 2, and $[H] = \begin{bmatrix} H_1 & 0 & H_2 & 0 \\ 0 & H_1 & 0 & H_2 \end{bmatrix}$ is the matrix of shape functions defined in Eq. (6.26). The integration can be performed in a closed form if the specified surface tractions (T_x and T_y) are a simple function of s. We will modify Eq. (6.32) to include all the six DOFs of the element and rewrite as

$$V^{(e)} = -\{\mathbf{q}^{(e)}\}^T h\int_{S_T}[\mathbf{N}(s)]^T\{\mathbf{T}(s)\}ds = -\{\mathbf{q}^{(e)}\}^T\{\mathbf{f}_T^{(e)}\} \tag{6.33}$$

We have used the complete shape function matrix in Eq. (6.33):

$$[\mathbf{N}] = \begin{bmatrix} \frac{l-s}{l} & 0 & \frac{s}{l} & 0 & 0 & 0 \\ 0 & \frac{l-s}{l} & 0 & \frac{s}{l} & 0 & 0 \end{bmatrix} \tag{6.34}$$

If the last expression in Eq. (6.33) is examined carefully, it is possible to note that the force vector $\{\mathbf{f}_T\}$ is a nodal force vector that is equivalent to the distributed force applied on the edge of the element. This is also called the work-equivalent nodal force vector. For a constant surface traction T_x and T_y, we can calculate the equivalent nodal force, as

$$\{\mathbf{f}_T^{(e)}\} = h\int_0^l[\mathbf{N}]^T\{\mathbf{T}\}ds = h\int_0^l\begin{bmatrix} (l-s)/l & 0 \\ 0 & (l-s)/l \\ s/l & 0 \\ 0 & s/l \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} T_x \\ T_y \end{Bmatrix}ds = \frac{hl}{2}\begin{Bmatrix} T_x \\ T_y \\ T_x \\ T_y \\ 0 \\ 0 \end{Bmatrix} \tag{6.35}$$

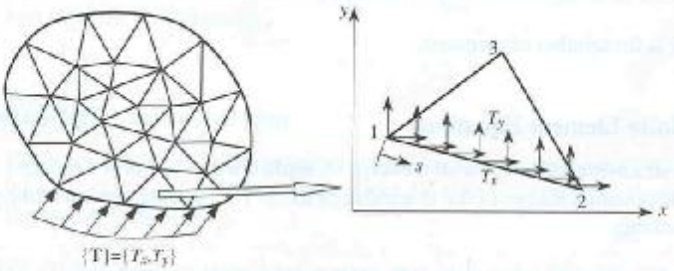


Figure 6.7 Applied surface traction along edge 1-2

For the uniform surface traction force, the equivalent nodal forces are obtained by simply dividing the total force equally between the two nodes on the edge.

The potential energy of distributed forces of all elements whose edge belongs to the traction boundary S_T must be assembled to build the global force vector of distributed forces:

$$V = -\sum_{e=1}^{NS} \{\mathbf{q}^{(e)}\}^T \{\mathbf{f}_T^{(e)}\} = -\{\mathbf{Q}_s\}^T \{\mathbf{F}_T\} \tag{6.36}$$

where NS is the number of elements whose edge belongs to S_T.

6.4.6 Potential Energy of Body Forces

The body forces are distributed over the entire element (e.g., centrifugal forces, gravitational forces, inertia forces, magnetic forces). For simplification of the derivation, let us assume that a constant body force $\mathbf{b} = \{b_x, b_y\}^T$ is applied to the whole element. The potential energy of body force becomes

$$\begin{aligned} V^{(e)} &= -h \iint_A \begin{bmatrix} u & v \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dA = -\{\mathbf{q}^{(e)}\}^T h \iint_A [\mathbf{N}]^T dA \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \\ &= -\{\mathbf{q}^{(e)}\}^T \{\mathbf{f}_b^{(e)}\} \end{aligned} \tag{6.37}$$

where

$$\{\mathbf{f}_b^{(e)}\} = \frac{hA}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \frac{hA}{3} \begin{Bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{Bmatrix} \tag{6.38}$$

The resultant of body forces is hAb_x in the x-direction and hAb_y in the y-direction. Equation (6.38) equally distributes these forces to the three nodes. Similar to the distributed force, $\{\mathbf{f}_B\}$ is the equivalent nodal force that corresponds to the constant body force.

The potential energy of body forces of all elements must be assembled to build the global force vector of body forces:

$$V = -\sum_{e=1}^{NE} \{\mathbf{q}^{(e)}\}^T \{\mathbf{f}_b^{(e)}\} = -\{\mathbf{Q}_s\}^T \{\mathbf{F}_B\} \tag{6.39}$$

where NE is the number of elements.

6.4.7 Global Finite Element Equations

Since the strain energy and potential energy of applied forces are now available, let us go back to the potential energy of the triangular element. The discrete version of the potential energy becomes

$$\Pi = U + V = \frac{1}{2}\{\mathbf{Q}_s\}^T [\mathbf{K}_s] \{\mathbf{Q}_s\} - \{\mathbf{Q}_s\}^T \{\mathbf{F}_N + \mathbf{F}_T + \mathbf{F}_B\} \tag{6.40}$$

The principle of minimum potential energy in Chapter 3 states that the structure is in equilibrium when the potential energy is minimum. Since the potential energy in Eq. (6.40) is a quadratic form the displacement vector $\{\mathbf{Q}_s\}$, we can differentiate Π to obtain

$$\frac{\partial \Pi}{\partial \{\mathbf{Q}_s\}} = 0 \Rightarrow [\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_N + \mathbf{F}_T + \mathbf{F}_B\} \tag{6.41}$$

The stationary condition of the potential energy yields the global finite element matrix equations.

The assembled structural stiffness matrix $[\mathbf{K}_s]$ is singular due to the presence of rigid body motion. After constructing the global matrix equation, the boundary conditions are applied by removing those DOFs that are fixed or prescribed. After imposing the boundary condition, the global stiffness matrix becomes non-singular and can be inverted to solve for the nodal displacements.

6.4.8 Calculation of Strains and Stresses

Once the nodal displacements are calculated, strains and stresses in individual elements can be calculated. First, the nodal displacement vector $\{\mathbf{q}^{(e)}\}$ for the element of interest needs to be extracted from the global displacement vector. Then, the strains and stresses in the element can be obtained from

$$\{\varepsilon\} = [\mathbf{B}]\{\mathbf{q}^{(e)}\} \tag{6.42}$$

and

$$\{\sigma\} = [\mathbf{C}]\{\varepsilon\} = [\mathbf{C}][\mathbf{B}]\{\mathbf{q}^{(e)}\} \tag{6.43}$$

where $[\mathbf{C}] = [\mathbf{C}_\sigma]$ for the plane stress problems and $[\mathbf{C}] = [\mathbf{C}_\varepsilon]$ for the plane strain problems.

As discussed before, stress and strain are constant within an element because the matrixes [B] and [C] are constant. This property can cause difficulties in interpreting the results of the finite element analysis. When two adjacent elements have different stress values, it is difficult to determine the stress value at the interface. Such discontinuity is not caused by the physics of the problem, but by the inability of the triangular element in describing the continuous change of stresses across element boundary. In fact, most finite elements cannot maintain continuity of stresses across the element boundary. Most programs average the stress at the element boundaries in order to make the stress look continuous. However, as we refine the model using smaller size elements, this discontinuity can be reduced. The following example illustrates discontinuity of stress and strain between two adjacent CST elements.

EXAMPLE 6.2 ***Cantilevered Plate***

Consider a cantilevered plate as shown in Figure 6.8. The plate has the following properties: $h = 0.1$ in, $E = 30 \times 10^6$ psi, and $\nu = 0.3$. Model the plate using two CST elements to determine the displacements and stresses.

SOLUTION This problem can be modeled as plane stress because the thickness of the plate is small compared to the other dimensions.

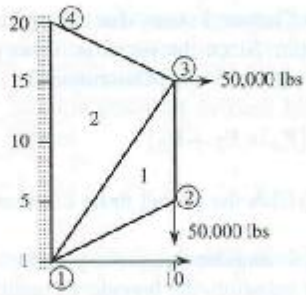


Figure 6.8 Cantilevered Plate

(1) Element 1: Nodes 1-2-3

Using nodal coordinates, we can calculate the constants defined in (6.18) as

$$\begin{array}{lll} x_1 = 0,\ y_1 = 0 & x_2 = 10,\ y_2 = 5 & x_3 = 10,\ y_3 = 15 \\ b_1 = y_2 - y_3 = -10 & b_2 = y_3 - y_1 = 15 & b_3 = y_1 - y_2 = -5 \\ c_1 = x_3 - x_2 = 0 & c_2 = x_1 - x_3 = -10 & c_3 = x_2 - x_1 = 10 \end{array}$$

In addition, from the geometry of the element, the area of the triangle $A_1 = 0.5 \times 10 \times 10 = 50$. The matrix $[\mathbf{B}]$ in Eq. (6.25) and $[\mathbf{C}_\sigma]$ can be written as

$$[\mathbf{B}] = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

$$= \frac{1}{100}\begin{bmatrix} -10 & 0 & 15 & 0 & -5 & 0 \\ 0 & 0 & 0 & -10 & 0 & 10 \\ 0 & -10 & -10 & 15 & 10 & -5 \end{bmatrix}$$

and

$$[\mathbf{C}_\sigma] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} = 3.297 \times 10^7 \begin{bmatrix} 1 & .3 & 0 \\ .3 & 1 & 0 \\ 0 & 0 & .35 \end{bmatrix}$$

Using the above two matrices, the element stiffness matrix can be obtained as

$$[\mathbf{k}^{(1)}] = hA[\mathbf{B}]^T[\mathbf{C}_\sigma][\mathbf{B}]$$

$$= 3.297 \times 10^6 \begin{bmatrix} .5 & 0. & -.75 & .15 & .25 & -.15 \\ & .175 & .175 & -.263 & -.175 & .088 \\ & & 1.3 & -.488 & -.55 & .313 \\ & & & .894 & .338 & -.631 \\ & \text{Symmetric} & & & .3 & -.163 \\ & & & & & .544 \end{bmatrix}$$

(2) Element 2: Nodes 1-3-4

By following similar procedures as for Element 1, the constants in Eq. (6.18) for Element 2 can be written as

$$\begin{array}{lll} x_1 = 0,\ y_1 = 0 & x_2 = 10,\ y_2 = 15 & x_3 = 0,\ y_3 = 20 \\ b_1 = y_2 - y_3 = -5 & b_2 = y_3 - y_1 = 20 & b_3 = y_1 - y_2 = -15 \\ c_1 = x_3 - x_2 = -10 & c_2 = x_1 - x_3 = 0 & c_3 = x_2 - x_1 = 10 \end{array}$$

The area of Element 2 is twice that of Element 1: $A_2 = 0.5 \times 20 \times 10 = 100$. The strain-displacement matrix [B] can be obtained as

$$[\mathbf{B}] = \frac{1}{200}\begin{bmatrix} -5 & 0 & 20 & 0 & -15 & 0 \\ 0 & -10 & 0 & 0 & 0 & 10 \\ -10 & -5 & 0 & 20 & 10 & -15 \end{bmatrix}$$

By following the same procedure, the stiffness matrix for Element 2 can be computed as

$$[\mathbf{k}^{(2)}] = 3.297 \times 10^6 \begin{bmatrix} .15 & .081 & -.25 & -.175 & .1 & .094 \\ & .272 & -.15 & -.088 & .069 & -.184 \\ & & 1. & 0. & -.75 & .15 \\ & & & .35 & .175 & -.263 \\ & & & & .65 & -.244 \\ & \text{Symmetric} & & & & .447 \end{bmatrix}$$

(3) Global finite element matrix equations

The two element stiffness matrices are assembled to form the global stiffness matrix. Since there are four nodes, the model has eight DOFs; each node has two DOFs. Thus, the global matrix has a dimension of 8×8. After assembly, the global matrix equation can be written as

$$3.297 \times 10^6 \begin{bmatrix} .65 & .081 & -.75 & .15 & .0 & -.325 & .1 & .094 \\ & .447 & .175 & -.263 & -.325 & .0 & .069 & -.184 \\ & & 1.3 & .488 & -.55 & .313 & 0 & .0 \\ & & & .894 & .338 & -.631 & 0 & .0 \\ & & & & 1.3 & -.163 & -.75 & .15 \\ & & & & & .894 & .175 & -.263 \\ & & & & & & .65 & -.244 \\ & \text{Symmetric} & & & & & & .447 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} R_{x1} \\ R_{y1} \\ 0 \\ -50,000 \\ 50,000 \\ 0 \\ R_{x4} \\ R_{y4} \end{Bmatrix}$$

where R_{x1}, R_{y1}, R_{x4}, and R_{y4} are unknown reaction forces at nodes 1 and 4.

(4) Applying boundary conditions

The displacement boundary conditions are: $u_1 = v_1 = u_4 = v_4 = 0$. Thus, we remove first, second, seventh, and eighth, rows and columns. After removing those rows and columns we obtain the following reduced matrix equation:

$$3.297 \times 10^6 \begin{bmatrix} 1.3 & -.488 & -.55 & .313 \\ & .894 & .338 & -.631 \\ & \text{Symmetric} & 1.3 & -.163 \\ & & & .894 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -50,000 \\ 50,000 \\ 0 \end{Bmatrix}$$

Note that the global stiffness matrix of the above equation is non-singular and, therefore, the unique solution can be obtained.

(5) Solution

The above matrix equation can be solved for unknown nodal displacements:

$$u_2 = -2.147 \times 10^{-5}$$
$$v_2 = -4.455 \times 10^{-2}$$
$$u_3 = 1.891 \times 10^{-2}$$
$$v_3 = -2.727 \times 10^{-2}$$

(6) Strain and stress in Element 1

After calculating the nodal displacements, strain and stress can be calculated at the element level. First, the displacements for those nodes that belong to the element need to be extracted from the global nodal displacement vector. Since nodes 1, 2, and 3 belong to Element 1, the nodal

displacements will be $\{\mathbf{q}\} = \{u_1, v_1, u_2, v_2, u_3, v_3\}^T = \{0, 0, -2.147 \times 10^{-3}, -4.455 \times 10^{-2}, 1.891 \times 10^{-2}, -2.727 \times 10^{-2}\}^T$. Then, strain in Eq. (6.25) can be calculated using $\{\varepsilon\} = [\mathbf{B}]\{\mathbf{q}\}$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{100}\begin{bmatrix} -10 & 0 & 15 & 0 & -5 & 0 \\ 0 & 0 & 0 & -10 & 0 & 10 \\ 0 & -10 & -10 & 15 & 10 & -5 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -2.147 \times 10^{-3} \\ -4.455 \times 10^{-2} \\ 1.891 \times 10^{-2} \\ -2.727 \times 10^{-2} \end{Bmatrix}$$

$$= \begin{Bmatrix} -1.268 \times 10^{-3} \\ 1.727 \times 10^{-3} \\ -3.212 \times 10^{-3} \end{Bmatrix}$$

The stresses in the element are obtained from Eq. (6.5)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = 3.297 \times 10^7 \begin{bmatrix} 1 & .3 & 0 \\ .3 & 1 & 0 \\ 0 & 0 & .35 \end{bmatrix}\begin{Bmatrix} -1.268 \times 10^{-3} \\ 1.727 \times 10^{-3} \\ -3.212 \times 10^{-3} \end{Bmatrix} = \begin{Bmatrix} -24,709 \\ 44,406 \\ -37,063 \end{Bmatrix} \text{psi}$$

(7) Strains and stresses in Element 2

Element 2 has Nodes 1, 3, and 4. Thus, the nodal displacements will be $\{\mathbf{q}\} = \{u_1, v_1, u_3, v_3, u_4, v_4\}^T = \{0, 0, 1.891 \times 10^{-2}, -2.727 \times 10^{-2}, 0, 0\}^T$. Using the element displacements, the strains and stresses in the element can be obtained as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{200}\begin{bmatrix} -5 & 0 & 20 & 0 & -15 & 0 \\ 0 & -10 & 0 & 0 & 0 & 10 \\ -10 & -5 & 0 & 20 & 10 & -15 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 1.891 \times 10^{-2} \\ -2.727 \times 10^{-2} \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1.891 \times 10^{-3} \\ 0 \\ -2.727 \times 10^{-3} \end{Bmatrix}$$

and

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = 3.297 \times 10^7 \begin{bmatrix} 1 & .3 & 0 \\ .3 & 1 & 0 \\ 0 & 0 & .35 \end{bmatrix}\begin{Bmatrix} 1.891 \times 10^{-3} \\ 0 \\ -2.727 \times 10^{-3} \end{Bmatrix} = \begin{Bmatrix} 62,354 \\ 18,706 \\ -31,469 \end{Bmatrix} \text{psi}$$

If the stresses in the two elements are examined, one can note that the stress value changes suddenly across the element boundary. For example, σ_{xx} in Element 1 is $-24,709$ psi, whereas in Element 2 it is 62,354 psi. Such a drastic change in stresses is an indicator that the finite element analysis results from the current model are not accurate and more elements are required.

From Example 6.2, we can conclude the following:

- Stresses are constant over the individual element.
- The solution is not accurate because there are large discontinuities in stresses across element boundaries.
- With only two elements, the mesh is very coarse and we obviously cannot expect very good results.

6.5 FOUR-NODE RECTANGULAR ELEMENT

6.5.1 Lagrange Interpolation for Rectangular Element

A rectangular element is composed of four nodes and eight DOFs (see Figure 6.9). It is a part of a plane solid that is composed of many rectangular elements. Each element shares its edge and two corner nodes with an adjacent element, except for those on the boundary. The four vertices of a rectangle are the nodes of that element, as shown in Figure 6.9. The first node of an element can arbitrarily be chosen. However, the sequence of Nodes 1, 2, 3, and 4 should be in the counter-clockwise direction. Each node has two displacements, u and v, respectively, in the x- and y-directions.

Since all edges are parallel to the coordinate directions, this element is not practical but useful as it is the basis for the quadrilateral element discussed in the following section. In addition, the behavior of the rectangular element is similar to that of the quadrilateral element. Shape functions can be calculated using procedures similar to that of CST element, but it is more instructive to use the Lagrange interpolation functions in x- and y-directions.

Consider the rectangular element in Figure 6.9. From the geometry, it is clear that $x_3 = x_2$, $y_4 = y_3$, $x_4 = x_1$, and $y_2 = y_1$. We will use a polynomial in x and y as the interpolation function. Since there are four nodes, we can apply four conditions and hence the polynomial should have four terms, as follows:

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \\ v &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy \end{aligned} \tag{6.44}$$

Let us calculate unknown coefficients α_i using the x-directional displacement u:

$$\begin{cases} u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 + \alpha_4 x_1 y_1 \\ u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 + \alpha_4 x_2 y_2 \\ u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 + \alpha_4 x_3 y_3 \\ u_4 = \alpha_1 + \alpha_2 x_4 + \alpha_3 y_4 + \alpha_4 x_4 y_4 \end{cases}$$

It is obvious that we need to invert the 4×4 matrix in order to calculate the interpolation coefficients.

Instead of matrix inversion method, we use the *Lagrange interpolation* method to interpolate u and v. The goal is to obtain the following expression:

$$u(x, y) = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{6.45}$$

where $N_1, \ldots, N_4$ are the interpolation functions. To do that, let us first consider displacement along edge 1-2 in Figure 6.9. Along edge 1-2, $y = y_1$ (constant); therefore shape functions must be functions of x only as shown below:

$$u_I(x, y_1) = [n_1(x) \quad n_2(x)] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{6.46}$$

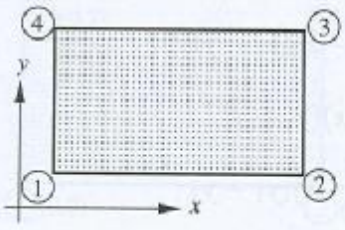


Figure 6.9 Four-node rectangular element

Using one-dimensional Lagrange interpolation formula the shape functions can be obtained as

$$n_1(x) = \frac{x - x_2}{x_1 - x_2}, \quad n_2(x) = \frac{x - x_1}{x_2 - x_1} \tag{6.47}$$

This is the same procedure that was used in Chapter 3. Next, since $y = y_3 = y_4$ along edge 4-3 in Figure 6.9, the displacement can be interpolated as

$$u_{II}(x, y_3) = [n_4(x) \quad n_3(x)] \begin{Bmatrix} u_4 \\ u_3 \end{Bmatrix} \tag{6.48}$$

Again from one-dimensional Lagrange interpolation formula, we have

$$n_4(x) = \frac{x - x_3}{x_4 - x_3}, \quad n_3(x) = \frac{x - x_4}{x_3 - x_4} \tag{6.49}$$

Equations (6.46) and (6.48) represent interpolation of displacements at the top and bottom of the element, respectively. So far, we have interpolated displacements in the x-direction only. Now, we can extend the interpolation in the y-direction between $u_I(x, y_1)$ and $u_{II}(x, y_3)$ using the same Lagrange interpolation method. By considering $u_I(x, y_1)$ and $u_{II}(x, y_3)$ as nodal displacements, we have the following interpolation formula:

$$u(x, y) = [n_1(y) \quad n_4(y)] \begin{Bmatrix} u_I(x, y_1) \\ u_{II}(x, y_3) \end{Bmatrix} \tag{6.50}$$

where

$$n_1(y) = \frac{y - y_4}{y_1 - y_4}, \quad n_4(y) = \frac{y - y_1}{y_4 - y_1} \tag{6.51}$$

are Lagrange interpolation in the y-direction. By substituting Eqs. (6.46) and (6.48) into Eq. (6.50), we have the following formula:

$$u(x, y) = [n_1(y) \quad n_4(y)] \begin{Bmatrix} [n_1(x) \quad n_2(x)] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ [n_4(x) \quad n_3(x)] \begin{Bmatrix} u_4 \\ u_3 \end{Bmatrix} \end{Bmatrix} \tag{6.52}$$

Thus,

$$u(x, y) = [n_1(x)n_1(y) \quad n_2(x)n_1(y) \quad n_3(x)n_4(y) \quad n_4(x)n_4(y)] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{6.53}$$

Comparing the above expression with Eq. (6.45), we can define shape functions. $N_1, \ldots, N_4$. In the rectangular element, it is enough to use the coordinates of two nodes, because $x_1 = x_4$, $y_1 = y_2$, etc. We will use the coordinates of Nodes 1 and 3. Using the property that the area of the element is $A = (x_3 - x_1)(y_3 - y_1)$, we obtain

$$\begin{cases} N_1 \equiv n_1(x)n_1(y) = \dfrac{1}{A}(x_3 - x)(y_3 - y) \\ N_2 \equiv n_2(x)n_1(y) = -\dfrac{1}{A}(x_1 - x)(y_3 - y) \\ N_3 \equiv n_3(x)n_4(y) = \dfrac{1}{A}(x_1 - x)(y_1 - y) \\ N_4 \equiv n_4(x)n_4(y) = -\dfrac{1}{A}(x_3 - x)(y_1 - y) \end{cases} \tag{6.54}$$

Note that the *shape functions* for rectangular elements are the product of Lagrange interpolations in the two coordinate directions. Let us discuss the properties of the shape functions. It can be easily verified that $N_1(x, y)$ is:

- 1 at Node 1 and 0 at other nodes
- linear function of x along edge 1-2 and linear function of y along edge 1-4 (bilinear interpolation)
- zero along edges 2-3 and 3-4

Other shape functions have similar behavior. Because of these characteristics, the i-th shape function is considered associated with Node i of the element.

To make the derivations simple, we rewrite the interpolation relation in Eq. (6.45) in matrix form. Let $\{\mathbf{u}\} = \{u, v\}^T$ be the displacement vector at any point (x, y). The interpolation can be written using the matrix notation by

$$\{\mathbf{u}\} \equiv \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

or,

$$\{\mathbf{u}\} = [\mathbf{N}]_{2\times 8}\{\mathbf{q}\}_{8\times 1} \tag{6.55}$$

Note that the dimension of the shape function matrix is 2×8.

EXAMPLE 6.3 ***Shape Functions of a Rectangular Element***

A rectangular element is shown in Figure 6.10. By substituting the numerical values of nodal coordinates into the above shape function formulas, the expressions for shape functions for this rectangular element can be obtained as

$$N_1 = \frac{(3-x)(2-y)}{6} \qquad N_2 = \frac{x(2-y)}{6}$$

$$N_3 = \frac{xy}{6} \qquad N_4 = \frac{y(3-x)}{6}$$

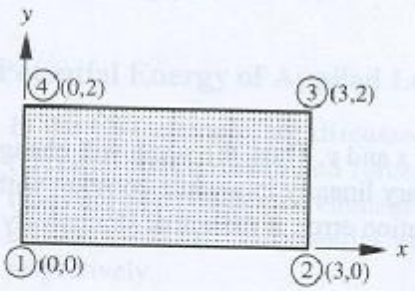
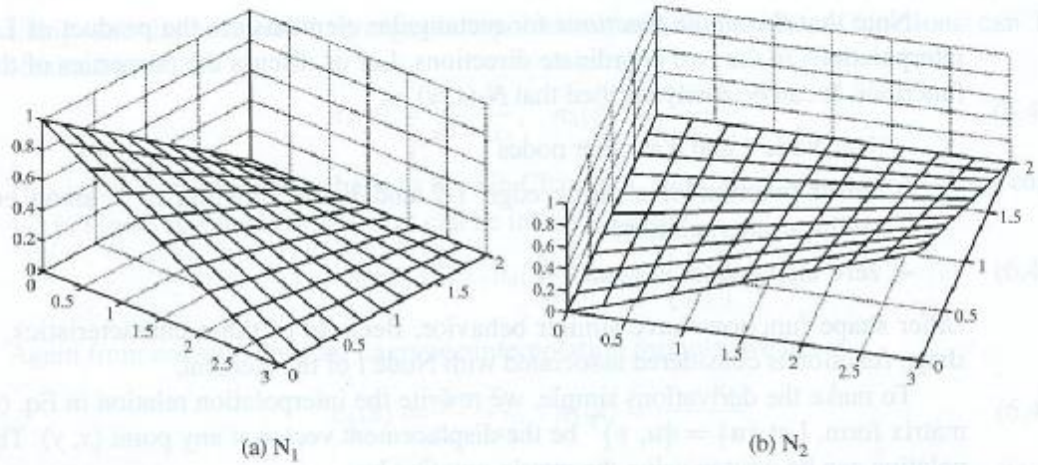


Figure 6.10 Four-node rectangular element



(a) N_1 (b) N_2

Figure 6.11 Three-dimensional surface plots of shape functions for a rectangular element

Three-dimensional plots of N_1 and N_2 are shown in Figure 6.11(a) and Figure 6.11(b).

Since the shape functions are given as function of x- and y-coordinates, we can use an approach similar to that of CST element to obtain the strain-displacement relations. Thus, the strain can be calculated by differentiating the shape function with respect to the coordinates. For example, ε_{xx} can be written as

$$\varepsilon_{xx} \equiv \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\sum_{i=1}^{4} N_i(x, y)u_i\right) = \sum_{i=1}^{4} \frac{\partial N_i}{\partial x} u_i \tag{6.56}$$

Note that u_1, u_2, u_3, and u_4 are nodal displacements and are independent of coordinate x. Thus, only the shape function is differentiated with respect to x. Similar calculation can be carried out for ε_{yy} and γ_{xy}. Then, we have

$$\{\varepsilon\} = \frac{1}{A}\begin{bmatrix} y-y_3 & 0 & y_3-y & 0 & y-y_1 & 0 & y_1-y & 0 \\ 0 & x-x_3 & 0 & x_1-x & 0 & x-x_1 & 0 & x_3-x \\ x-x_3 & y-y_3 & x_1-x & y_3-y & x-x_1 & y-y_1 & x_3-x & y_1-y \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\equiv [\mathbf{B}]\{\mathbf{q}\} \tag{6.57}$$

Note that the matrix [**B**] is a linear function of x and y. Thus, the strain will change linearly within the element. For example, ε_{xx} will vary linearly in y while constant with respect to x. Thus, the element will have an approximation error, if the actual strains vary in the x-direction.

6.5.2 Element Stiffness Matrix

The element stiffness matrix can be calculated from the strain energy of the element. By substituting for strains from Eq. (6.57) into the expression for strain energy in Eq. (6.9), we have

$$\begin{aligned} U^{(e)} &= \frac{h}{2}\iint_A \{\varepsilon\}^T[\mathbf{C}]\{\varepsilon\}dA^{(e)} \\ &= \frac{h}{2}\{\mathbf{q}^{(e)}\}^T \iint_A [\mathbf{B}]_{8\times 3}^T[\mathbf{C}]_{3\times 3}[\mathbf{B}]_{3\times 8}\, dA\{\mathbf{q}^{(e)}\} \\ &\equiv \frac{1}{2}\{\mathbf{q}^{(e)}\}^T[\mathbf{k}^{(e)}]_{8\times 8}\{\mathbf{q}^{(e)}\} \end{aligned} \tag{6.58}$$

where $[\mathbf{k}^{(e)}]$ is the element stiffness matrix. Calculation of the element stiffness matrix requires two-dimensional integration. We will discuss numerical integration in the next section. When the element is square and the problem is plane stress, analytical integration of the strain energy yields the following form of element stiffness matrix:

$$[\mathbf{k}^{(e)}] = \frac{Eh}{1-\nu^2}\times \begin{vmatrix} \frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} \\ \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} \\ -\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} \\ \frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} \\ \frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} \\ -\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} \\ \frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} \\ \frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6} \end{vmatrix} \tag{6.59}$$

It is interesting to note that the element stiffness matrix does not depend on the actual element dimensions but it is a function of material properties (E and ν) and thickness h for a square element.

The strain energy of the entire solid can be obtained using Eq. (6.29), which involves the assembly process.

6.5.3 Potential Energy of Applied Loads

In the CST element, we discussed three different types of applied loads: concentrated forces at nodes, distributed forces along element edges, and body force. The first two types are independent of element used. Thus, the same forms in Eqs. (6.31) and (6.36) can be used for the potential energies of concentrated force and distributed force, respectively.

In the case of the body force, the element shape functions are used to calculate equivalent nodal forces. When a constant body force $\mathbf{b} = \{b_x, b_y\}^T$ acts on a rectangular element, the potential energy of body force becomes

$$\begin{aligned} V^{(e)} &= -h \iint_A [u \quad v] \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dA = -\{\mathbf{q}^{(e)}\}^T h \iint_A [\mathbf{N}]^T dA \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \\ &\equiv \{\mathbf{q}^{(e)}\}^T \{\mathbf{f}_b^{(e)}\} \end{aligned} \tag{6.60}$$

where

$$\{\mathbf{f}_b^{(e)}\} = \frac{hA}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \frac{hA}{4} \begin{Bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{Bmatrix} \tag{6.61}$$

Equation (6.61) equally divides the total magnitude of the body force to the four nodes. $\{\mathbf{f}_b\}$ is the equivalent nodal force that corresponds to the constant body force. The potential energy of body forces of all elements must be assembled to build the global force vector of body forces as in Eq. (6.39).

Using the principle of minimum total potential energy in Eq. (6.41), a similar global matrix equation for the rectangular elements can be obtained. Applying boundary conditions and solving the matrix equations are identical to the CST element. After solving for nodal displacements, strains and stresses in each element can be calculated using Eqs. (6.42) and (6.43), respectively.

EXAMPLE 6.4 *Simple Shear Deformation of a Square Element*

A square element shown in Figure 6.12 is under a simple shear deformation. Material properties are given as $E = 10\,\text{GPa}$, $\upsilon = 0.25$, and thickness is $h = 0.1\,\text{m}$. When distributed force $f = 100\,\text{kN/m}^2$ is applied horizontally at the top edge, calculate stress and strain components. Compare the results with the exact solution.

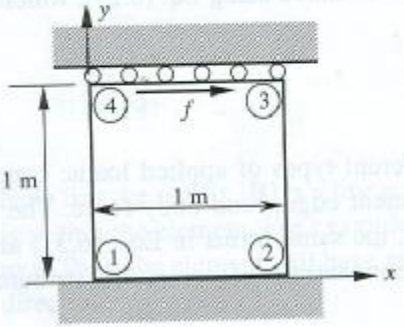


Figure 6.12 A square element under a simple shear condition

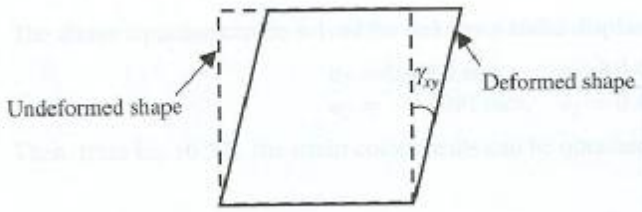


Figure 6.13 Simple shear deformation of a square element

SOLUTION Since the problem consists of one element, we do not need assembly process. The element has eight DOFs: $\{\mathbf{Q}_s\} = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}^T$. From the boundary condition given in Figure 6.12, only two DOFs are non-zero: u_3 and u_4. Thus, from the element stiffness matrix given in Eq. (6.59), all fixed DOFs are deleted to obtain

$$[\mathbf{K}] = \frac{Eh}{1-\nu^2}\begin{bmatrix} \frac{3-\nu}{6} & \frac{3+\nu}{12} \\ \frac{3+\nu}{12} & \frac{3-\nu}{6} \end{bmatrix} \begin{matrix} u_3 \\ u_4 \end{matrix} = 10^8 \begin{bmatrix} 4.88 & -2.88 \\ -2.88 & 4.88 \end{bmatrix} \begin{matrix} u_3 \\ u_4 \end{matrix}$$

The total distributed load of 10,000 N at the top edge will be equally divided into two nodes: 4 and 3. Thus, the global matrix equation becomes

$$10^8 \begin{bmatrix} 4.88 & -2.88 \\ -2.88 & 4.88 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 5{,}000 \\ 5{,}000 \end{Bmatrix}$$

The above equation can be solved for unknown nodal displacements, as $u_3 = u_4 = 0.025$ mm. Then, from (6.57), the strain components can be obtained, as

$$\{\varepsilon\} = \begin{bmatrix} y-1 & 0 & 1-y & 0 & y & 0 & -y & 0 \\ 0 & x-1 & 0 & -x & 0 & x & 0 & 1-x \\ x-1 & y-1 & -x & 1-y & x & y & 1-x & -y \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2.5\times10^{-5} \\ 0 \\ 2.5\times10^{-5} \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 2.5\times10^{-5} \end{Bmatrix}$$

Note that the shear strain is the only non-zero strain. Thus, the rectangular element can accurately represent the simple shear condition. Figure 6.13 shows the deformed shape of the solid. Note that the deformation is magnified for the illustration purpose.

Using the stress-strain relation in Eq. (6.5) for plane stress, the stress components can be obtained, as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{10^{10}}{1-0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 2.5\times10^{-5} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10^5 \end{Bmatrix} \text{Pa}$$

Since distributed force $f = 10\,\text{kN/m}^2$ is applied at the top edge, the above shear stress is exact.

EXAMPLE 6.5 *Pure Bending Deformation of a Square Element*

A pure bending condition can be achieved by applying a couples in the case of a beam (see Chapter 4). For the plane solid, the effect of a couple can be achieved by applying equal forces in opposite

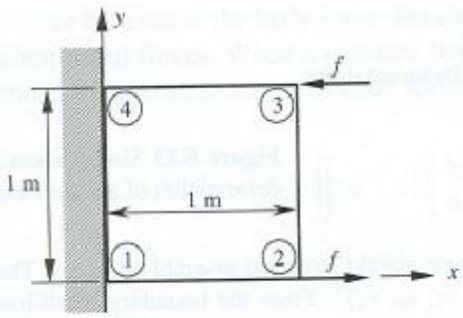


Figure 6.14 A square element under pure bending condition

directions. A square element shown in Figure 6.14 is under a pure bending condition. Material properties are given as $E = 10\,\text{GPa}$, $\nu = 0.25$, and thickness is $h = 0.1$ m. When an equal and opposite force $f = 100\,\text{kN}$ is applied at Nodes 2 and 3, calculate stress and strain components. Compare the results with the exact solutions from the beam theory.

SOLUTION

(a) Analytical solution: If we consider the above plane solid as a cantilevered beam, the moment of inertia $I = 8.333 \times 10^{-3}\,\text{m}^4$ and the applied couple $M = 100\,\text{kN} \cdot \text{m}$. Thus, the maximum stress will occur at the bottom edge with the magnitude of

$$(\sigma_{xx})_{\max} = -\frac{M\left(-\frac{\text{L}}{2}\right)}{I} = 6.0\,\text{MPa}$$

where L is the length of the element.The minimum stress will occur at the top edge with the same magnitude, but in compression. Since the stress varies linearly along the y-coordinate, we have

$$\sigma_{xx} = 6.0(1 - 2y)\,\text{MPa}$$

All other stress components are zero.

(b) Finite element solution: Since we use only one element, we do not need assembly process. The element has eight DOFs: $\{\mathbf{Q}_s\} = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}^T$. From the boundary condition given in Figure 6.14, only four DOFs are non-zero: u_2, v_2, u_3, and v_3. Thus, from the stiffness matrix of a square element given in Eq. (6.59), all fixed DOFs are deleted to obtain

$$[\mathbf{K}] = \frac{Eh}{1-\nu^2}\begin{bmatrix} \frac{3-\nu}{6} & \frac{1+\nu}{8} & \frac{\nu}{6} & \frac{-1+3\nu}{8} \\ \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{3+\nu}{12} \\ \frac{\nu}{6} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & \frac{1+\nu}{8} \\ \frac{-1+3\nu}{8} & \frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{3-\nu}{6} \end{bmatrix}$$

$$= 10^8 \begin{bmatrix} 4.89 & -1.67 & 0.44 & -0.33 \\ -1.67 & 4.89 & 0.33 & -2.89 \\ 0.44 & 0.33 & 4.89 & 1.67 \\ -0.33 & -2.89 & 1.67 & 4.89 \end{bmatrix} \begin{matrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix}$$

Using the applied nodal forces, the global matrix equation becomes

$$10^8 \begin{bmatrix} 4.89 & -1.67 & 0.44 & -0.33 \\ -1.67 & 4.89 & 0.33 & -2.89 \\ 0.44 & 0.33 & 4.89 & 1.67 \\ -0.33 & -2.89 & 1.67 & 4.89 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 100{,}000 \\ 0 \\ -100{,}000 \\ 0 \end{Bmatrix}$$

The above equation can be solved for unknown nodal displacements as

$$u_2 = 0.4091\ \text{mm}, \quad v_2 = 0.4091\ \text{mm}$$
$$u_3 = -0.4091\ \text{mm}, \quad v_3 = 0.4091\ \text{mm}$$

Then, from Eq. (6.57), the strain components can be obtained as

$$\{\varepsilon\} = \begin{bmatrix} y-1 & 0 & 1-y & 0 & y & 0 & -y & 0 \\ 0 & x-1 & 0 & -x & 0 & x & 0 & 1-x \\ x-1 & y-1 & -x & 1-y & x & y & 1-x & -y \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.4091 \\ 0.4091 \\ -0.4091 \\ 0.4091 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-3}$$

$$= \begin{Bmatrix} 0.4091 \times 10^{-3}(1-2y) \\ 0 \\ 0.4091 \times 10^{-3}(1-2x) \end{Bmatrix}$$

Using the stress-strain relation in Eq. (6.5) for plane stress, the stress components can be obtained as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{10^{10}}{1-0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \begin{Bmatrix} 0.4091 \times 10^{-3}(1-2y) \\ 0 \\ 0.4091 \times 10^{-3}(1-2x) \end{Bmatrix}$$

$$= \begin{Bmatrix} 4.364(1-2y) \\ 1.091(1-2y) \\ 1.636(1-2x) \end{Bmatrix} \text{MPa}$$

The deformed shape of the element is shown in the figure below.

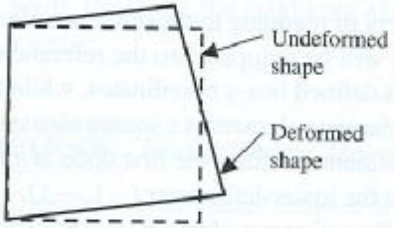
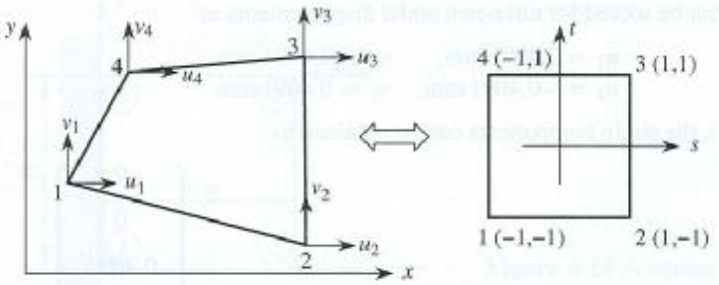


Figure 6.15 Pure bending deformation of a square element

In a plane solid, the applied couple produces a curvature, but the rectangular element is unable to produce deformation corresponding to the curvature because the displacement can only change linearly within the element. The rectangular shape deforms to the trapezoidal shape, and as a result, non-zero shear stress is produced. Note that the maximum stress $(\sigma_{xx})_{\max}$ is only 73% (4.364/6.0) of the exact solution. In addition, σ_{yy} and τ_{xy} have non-zero values. The applied couple is supported by other stress components, σ_{yy} and τ_{xy}, and as a result the element shows smaller $(\sigma_{xx})_{\max}$. In a sense, the element shows a *stiff* behavior.

6.6 FOUR-NODE ISO-PARAMETRIC QUADRILATERAL ELEMENT

As discussed in Section 6.5, the rectangular element is limited in practical applications due to its inability to represent irregular geometries. The four-node quadrilateral element, shown in Figure 6.16, can overcome such limitations. The four-node iso-parametric finite element is one of the most commonly used elements in engineering applications.



(a) Physical element (b) Reference element

Figure 6.16 Four-node quadrilateral element for plane solids

The element consists of four nodes and two DOFs at each node. Since the geometry of the element is irregular, it is convenient to introduce a *reference element* and use a mapping relation between the physical element and the reference element. The term *iso-parametric* comes from the fact that the same interpolation scheme is used for interpolating both displacement and geometry.

6.6.1 Iso-parametric Mapping

The physical element in Figure 6.16 is a general quadrilateral shape. However, all interior angles should be less than 180 degrees. The order of node numbers is the same as that of the rectangular element: starting from one corner and moving in the counter-clockwise direction. Each node has two DOFs: u and v. Thus, the element has a total of eight DOFs.

Since different elements have different shapes, it would not be a trivial task if the interpolation functions need to be developed for individual elements. The interpolation functions must satisfy the inter-element displacement compatibility condition discussed earlier in the context of triangular elements. Instead, the concept of mapping to the reference element will be used. The physical element in Figure 6.16(a) will be mapped into the reference element shown in Figure 6.16(b). The physical element is defined in x-y coordinates, while the reference element is defined in s-t coordinates. The reference element is a square element and has the center at the origin. Although the physical element can have the first node at any corner, the reference element always has the first node at the lower-left corner $(-1, -1)$.

The interpolation functions are defined in the reference element so that different elements have the same interpolation function. The only difference is the mapping relation between the two elements. Since the reference element is of square shape, the Lagrange interpolation for rectangular elements can be used. Using (6.54), the interpolation or shape functions can be written in s-t coordinates as

$$\begin{cases} N_1(s,t) = \dfrac{1}{4}(1-s)(1-t) \\ N_2(s,t) = \dfrac{1}{4}(1+s)(1-t) \\ N_3(s,t) = \dfrac{1}{4}(1+s)(1+t) \\ N_4(s,t) = \dfrac{1}{4}(1-s)(1+t) \end{cases} \tag{6.62}$$

Since the above shape functions are Lagrange interpolation functions, they satisfy the same properties as those of the rectangular element. Thus, N_1 is equal to unity at Node 1 and zero at other nodes, etc.

Note that in the CST and rectangular elements, the shape functions are used to interpolate displacements within the element. In the quadrilateral element, the shape functions are also used for mapping between the physical element and the reference element. The quadrilateral element is defined by the coordinates of four corner nodes. These four corner nodes are mapped into the four corner nodes of the reference element. In addition, every point in the physical element is also mapped into a point in the reference element. The mapping relation is one-to-one such that every point in the reference element also has a mapped point in the physical element. Thus a physical point (x, y) is a function of reference point (s, t). A relation between (x, y) and (s, t) can be derived using the same shape functions as

$$x(s,t) = [N_1(s,t) \quad N_2(s,t) \quad N_3(s,t) \quad N_4(s,t)] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$
$$y(s,t) = [N_1(s,t) \quad N_2(s,t) \quad N_3(s,t) \quad N_4(s,t)] \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \tag{6.63}$$

It can be easily checked that at Node 1, for example, $(s, t) = (-1, -1)$ and $N_1 = 1$, $N_2 = N_3 = N_4 = 0$. Thus, we have $x(-1, -1) = x_1$ and $y(-1, -1) = y_1$, i.e., Node 1 in the physical element is mapped into Node 1 in the reference element. The above mapping relation is called *iso-parametric mapping* because the same shape functions are used for interpolating geometry as well as displacements.

The above mapping relation is explicit in terms of x and y, which means that when s and t are given, x and y can be calculated explicitly. The reverse relation is not straightforward. However, the following example explains how s and t can be calculated for a given x and y.

EXAMPLE 6.6 *Iso-Parametric Mapping*

Consider a quadrilateral element of the trapezoidal shape shown in Figure 6.17. Using the iso-parametric mapping method calculate: (a) the physical coordinates of point A (0.5, 0.5), and (b) the reference coordinate of point B (1, 2).

SOLUTION

(a) At point A, $(s, t) = (0.5, 0.5)$. The values of the shape functions at A are

$$N_1\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{1}{16}, \quad N_2\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{3}{16}, \quad N_3\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{9}{16}, \quad N_4\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{3}{16}$$

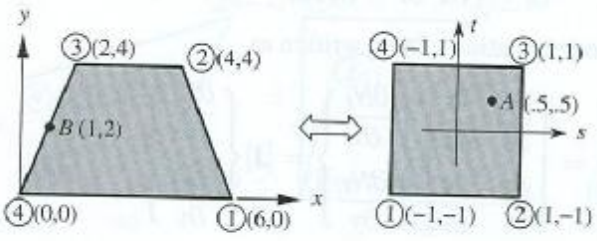


Figure 6.17 Mapping of a quadrilateral element

Thus, the physical coordinate becomes

$$x\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\sum_{I=1}^{4}N_I\left(\tfrac{1}{2},\tfrac{1}{2}\right)x_I=\frac{1}{16}\cdot 6+\frac{3}{16}\cdot 4+\frac{9}{16}\cdot 2+\frac{3}{16}\cdot 0=2.25$$

$$y\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\sum_{I=1}^{4}N_I\left(\tfrac{1}{2},\tfrac{1}{2}\right)y_I=\frac{1}{16}\cdot 0+\frac{3}{16}\cdot 4+\frac{9}{16}\cdot 4+\frac{3}{16}\cdot 0=3$$

Thus, the reference point $(s, t) = (0.5, 0.5)$ is mapped into the physical point $(x, y) = (2.25, 3.0)$.

(b) At point B, $(x, y) = (1, 2)$. From the iso-parametric mapping relation, we have

$$\begin{aligned}x=1=\sum_{I=1}^{4}N_I(s,t)x_I&=\tfrac{1}{4}(1-s)(1-t)\cdot 6+\tfrac{1}{4}(1+s)(1-t)\cdot 4\\&\quad+\tfrac{1}{4}(1+s)(1+t)\cdot 2+\tfrac{1}{4}(1-s)(1+t)\cdot 0\\&=st-2t+3\end{aligned}$$

$$\begin{aligned}y=2=\sum_{I=1}^{4}N_I(s,t)y_I&=\tfrac{1}{4}(1-s)(1-t)\cdot 0+\tfrac{1}{4}(1+s)(1-t)\cdot 4\\&\quad+\tfrac{1}{4}(1+s)(1+t)\cdot 4+\tfrac{1}{4}(1-s)(1+t)\cdot 0\\&=2+2s\end{aligned}$$

From the above two relations, we obtain $(s, t) = (0, 1)$. Note that the above results will not be the same, if the sequence of node numbers in the physical element is changed.

6.6.2 Jacobian of Mapping

The idea of using the reference element is convenient because it is unnecessary to build different shape functions for different elements. The same shape functions can be used for all elements. However, it has its own drawbacks. The strain energy in the plane solid element requires the derivative of displacement, i.e., strains. As can be seen in Eq. (6.25), the strains are defined as derivatives of displacements. In the case of CST and rectangular elements, the shape functions could be differentiated directly because the nodal displacements are explicit functions of x and y. For those elements, the derivatives of the shape functions can be easily obtained because they are defined as a function of physical coordinates (x, y). However, in the case of the quadrilateral element, the shape functions are defined in the reference coordinates. Thus, differentiation with respect to the physical coordinates is not straightforward. In this case, we use a Jacobian relation and the chain rule of differentiation. From the fact that $s = s(x, y)$ and $t = t(x, y)$, we can write the derivatives of N_I as follows:

$$\frac{\partial N_I}{\partial s}=\frac{\partial N_I}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial N_I}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial N_I}{\partial t}=\frac{\partial N_I}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial N_I}{\partial y}\frac{\partial y}{\partial t}$$

Using the matrix form, the above equation can be written as

$$\begin{Bmatrix}\dfrac{\partial N_I}{\partial s}\\[2ex]\dfrac{\partial N_I}{\partial t}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s}\\[2ex]\dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial N_I}{\partial x}\\[2ex]\dfrac{\partial N_I}{\partial y}\end{Bmatrix}=[\mathbf{J}]\begin{Bmatrix}\dfrac{\partial N_I}{\partial x}\\[2ex]\dfrac{\partial N_I}{\partial y}\end{Bmatrix}\tag{6.64}$$

where [J] is the *Jacobian matrix* and its determinant is called the Jacobian. By inverting the Jacobian matrix, the desired derivatives with respect to x and y can be obtained:

$$\begin{Bmatrix} \frac{\partial N_I}{\partial x} \\ \frac{\partial N_I}{\partial y} \end{Bmatrix} = [\mathbf{J}]^{-1} \begin{Bmatrix} \frac{\partial N_I}{\partial s} \\ \frac{\partial N_I}{\partial t} \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \frac{\partial y}{\partial t} & -\frac{\partial y}{\partial s} \\ -\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_I}{\partial s} \\ \frac{\partial N_I}{\partial t} \end{Bmatrix} \tag{6.65}$$

where $|\mathbf{J}|$ is the *Jacobian* and is defined by

$$|\mathbf{J}| = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \tag{6.66}$$

Since iso-parametric mapping is used, the above Jacobian can be obtained by differentiating the relation in Eq. (6.63) with respect to s and t. For example,

$$\frac{\partial x}{\partial s} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial s} x_I = \frac{1}{4}(-x_1 + x_2 + x_3 - x_4) + \frac{t}{4}(x_1 - x_2 + x_3 - x_4)$$

$$\frac{\partial x}{\partial t} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial t} x_I = \frac{1}{4}(-x_1 - x_2 + x_3 + x_4) + \frac{s}{4}(x_1 - x_2 + x_3 - x_4)$$

A similar expression can be obtained for $\partial y/\partial s$ and $\partial y/\partial t$ by replacing x_i with y_i. Note that $\partial x/\partial s$ is the function of t only, while $\partial x/\partial t$ is the function of s only.

As seen from Eq. (6.65), the derivative of the shape function cannot be obtained if the Jacobian is zero anywhere in the element. In fact, the mapping relation between (x, y) and (s, t) is not valid if the Jacobian is zero or negative anywhere in the element $(-1 \le s, t \le 1)$.

The Jacobian plays an important role in evaluating the validity of mapping as well as the quality of the quadrilateral element. The fundamental requirement is that every point in the reference element should be mapped into the interior of the physical element, and vice versa. When an interior point in (s, t) coordinates is mapped into an exterior point in the (x, y) coordinates, the Jacobian becomes negative. If multiple points in (s, t) coordinates are mapped into a single point in (x, y) coordinates, the Jacobian becomes zero at that point. Thus, it is important to maintain the element shape so that the Jacobian is positive everywhere in the element.

EXAMPLE 6.7 *Jacobian of Mapping*

Check the validity of iso-parametric mapping for the two elements shown in Figure 6.18.

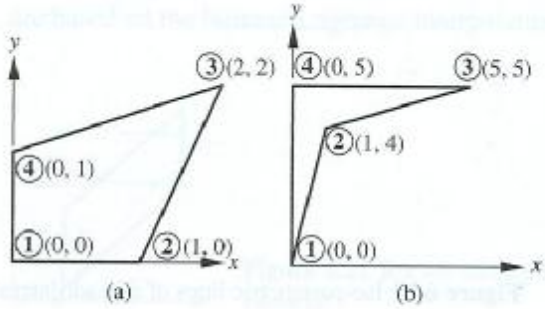


Figure 6.18 Four-node quadrilateral elements

SOLUTION

(a) Nodal coordinates:

$$x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 0$$
$$y_1 = 0, y_2 = 0, y_3 = 2, y_4 = 1$$

– Iso-parametric mapping:

$$x = \sum_{I=1}^{4} N_I x_I = N_2 + 2N_3 = \frac{1}{4}(3 + 3s + t + st)$$
$$y = \sum_{I=1}^{4} N_I y_I = 2N_3 + N_4 = \frac{1}{4}(3 + s + 3t + st)$$

– Jacobian:

$$[\mathbf{J}] = \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3+t & 1+t \\ 1+s & 3+s \end{bmatrix}$$

$$|\mathbf{J}| = \frac{1}{4}[(3+t)(3+s) - (1+t)(1+s)] = \frac{1}{2} + \frac{1}{8}s + \frac{1}{8}t$$

Thus, it is clear that $|\mathbf{J}| > 0$ for $-1 \le s \le 1$ and $-1 \le t \le 1$. Figure 6.19 shows constant s and t lines. Since all lines are within the element boundary, the mapping is valid.

(b) Nodal coordinates:

$$x_1 = 0, x_2 = 1, x_3 = 5, x_4 = 0$$
$$y_1 = 0, y_2 = 4, y_3 = 5, y_4 = 5$$

– Iso-parametric mapping:

$$x = \sum_{I=1}^{4} N_I x_I = \frac{1}{2}(1 + s)(3 + 2t)$$
$$y = \sum_{I=1}^{4} N_I y_I = \frac{1}{2}(7 + 2s + 3t - 2st)$$

– Jacobian:

$$|\mathbf{J}| = \frac{1}{4}(5 - 10s + 10t)$$

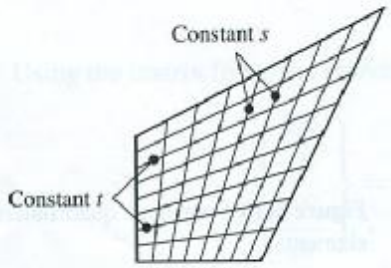


Figure 6.19 Iso-parametric lines of a quadrilateral element

شماره ثبت ۹۵۴۰
کتابخانه دانشکده مهندسی مکانیک

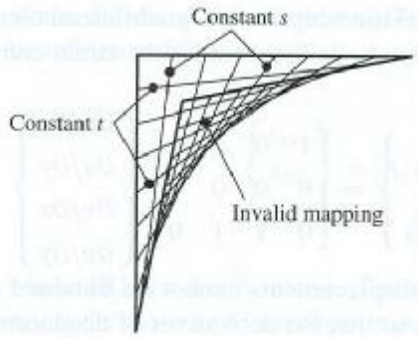


Figure 6.20 An example of invalid mapping

Note that $|\mathbf{J}| = 0$ at $5 - 10s + 10t = 0$; i.e., $s - t = 1/2$. The mapping illustrated in Figure 6.20 clearly shows that the mapping is invalid. Some points in the reference element are mapped into the outside of the physical element.

In the practical sense, maintaining positive Jacobian is not enough because of the numerical nature. For example, when the Jacobian is small, i.e., $|\mathbf{J}| \ll 1$, calculation of stress and strain is not accurate and the integration of the strain energy will losen its accuracy. The small value of the Jacobian occurs when the element shape is far from the rectangular one. To avoid problems due to badly shaped elements, it is recommended that the inside angles in quadrilateral elements be $> 15°$ and $< 165°$, as illustrated in Figure 6.21.

6.6.3 Interpolation of Displacements and Strains

As we explained in iso-parametric mapping above, the same shape functions are used for interpolating displacements. Similar to the rectangular element the quadrilateral element also has eight DOFs. Then the displacements within the element can be interpolated as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = [\mathbf{N}]\{\mathbf{q}\} \tag{6.67}$$

where the same shape functions in Eq. (6.62) are used for interpolation. The difference between the previous two elements (CST and rectangular elements) and the quadrilateral element is that the interpolation is done in the reference coordinates (s, t). However, the behavior of the element is similar to that of the rectangular element because both of them are based on the bilinear Lagrange interpolation

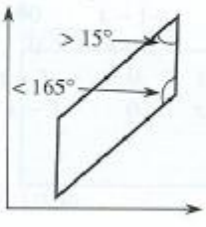


Figure 6.21 Recommended ranges of internal angles in a quadrilateral element

Now we derive the strain-displacement relationship for the quadrilateral element. To make the following matrix operation convenient, we first reorder the strain components into the derivatives of displacements, as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{Bmatrix}$$

As we discussed above, the derivatives of displacements cannot be obtained directly. Instead, we use the inverse Jacobian relation so that the derivatives of displacements are written in terms of the reference coordinates. Thus, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \dfrac{\partial y}{\partial t} & -\dfrac{\partial y}{\partial s} \\ -\dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial s} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial s} \\ \dfrac{\partial u}{\partial t} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \dfrac{\partial y}{\partial t} & -\dfrac{\partial y}{\partial s} \\ -\dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial s} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial s} \\ \dfrac{\partial v}{\partial t} \end{Bmatrix}$$

Writing the two equations together, we have

$$\begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \partial y/\partial t & -\partial y/\partial s & 0 & 0 \\ -\partial x/\partial t & \partial x/\partial s & 0 & 0 \\ 0 & 0 & \partial y/\partial t & -\partial y/\partial s \\ 0 & 0 & -\partial x/\partial t & \partial x/\partial s \end{bmatrix} \begin{Bmatrix} \partial u/\partial s \\ \partial u/\partial t \\ \partial v/\partial s \\ \partial v/\partial t \end{Bmatrix}$$

The strains can now be expressed as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \partial y/\partial t & -\partial y/\partial s & 0 & 0 \\ -\partial x/\partial t & \partial x/\partial s & 0 & 0 \\ 0 & 0 & \partial y/\partial t & -\partial y/\partial s \\ 0 & 0 & -\partial x/\partial t & \partial x/\partial s \end{bmatrix} \begin{Bmatrix} \partial u/\partial s \\ \partial u/\partial t \\ \partial v/\partial s \\ \partial v/\partial t \end{Bmatrix}$$

$$\equiv [\mathbf{A}] \begin{Bmatrix} \partial u/\partial s \\ \partial u/\partial t \\ \partial v/\partial s \\ \partial v/\partial t \end{Bmatrix}$$

where $[\mathbf{A}]$ is a 3×4 matrix. The derivatives of the displacements with respect to s and t can be obtained by differentiating $u(s,t)$ and $v(s,t)$ in Eq. (6.67), which involves the derivatives of the shape functions:

$$\begin{Bmatrix} \partial u/\partial s \\ \partial u/\partial t \\ \partial v/\partial s \\ \partial v/\partial t \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} -1+t & 0 & 1-t & 0 & 1+t & 0 & -1-t & 0 \\ -1+s & 0 & -1-s & 0 & 1+s & 0 & 1-s & 0 \\ 0 & -1+t & 0 & 1-t & 0 & 1+t & 0 & -1-t \\ 0 & -1+s & 0 & -1-s & 0 & 1+s & 0 & 1-s \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\equiv [\mathbf{G}]\{\mathbf{q}\}$$

where the dimension of matrix [G] is 4×8. The strain-displacement matrix [B] can now be written as follows:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = [\mathbf{A}] \begin{Bmatrix} \partial u/\partial s \\ \partial u/\partial t \\ \partial v/\partial s \\ \partial v/\partial t \end{Bmatrix} = [\mathbf{A}][\mathbf{G}]\{\mathbf{q}\} \equiv [\mathbf{B}]\{\mathbf{q}\} \tag{6.68}$$

where [B] is a 3×8 matrix. The explicit expression of [B] is not readily available because the matrix [A] involves inverse of Jacobian matrix. However, for given reference coordinate (s, t), it can be calculated using Eq. (6.68). Note that the strain-displacement matrix [B] is not constant as in CST elements. Thus, the strains and stresses within an element vary as a function of s and t coordinates.

EXAMPLE 6.8 ***Interpolation using Quadrilateral Element***

For a rectangular element shown in Figure 6.22, displacements at four nodes are given by $\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{0.0, 0.0, 1.0, 0.0, 2.0, 1.0, 0.0, 2.0\}$. Calculate displacement and strain at point $(s, t) = (1/3, 0)$.

SOLUTION When the reference coordinate $(s, t) = (1/3, 0)$, the shape functions become

$$N_1 = \frac{1}{6}, \qquad N_2 = \frac{1}{3}, \qquad N_3 = \frac{1}{3}, \qquad N_4 = \frac{1}{6}$$

Using Eq. (6.67), the displacements can be interpolated, as

$$\begin{cases} u = \sum_{I=1}^{4} N_I u_I = \frac{1}{6} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{6} \cdot 0 = 1 \\ v = \sum_{I=1}^{4} N_I v_I = \frac{1}{6} \cdot 0 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{6} \cdot 2 = \frac{2}{3} \end{cases}$$

In order to calculate strains, we need the derivatives of the shape functions. First, we calculate the derivatives with respect to the reference coordinates, as

$$\begin{cases} \frac{\partial N_1}{\partial s} = -\frac{1}{4}(1-t) = -\frac{1}{4} \\ \frac{\partial N_2}{\partial s} = \frac{1}{4}(1-t) = \frac{1}{4} \\ \frac{\partial N_3}{\partial s} = \frac{1}{4}(1+t) = \frac{1}{4} \\ \frac{\partial N_4}{\partial s} = -\frac{1}{4}(1+t) = -\frac{1}{4} \end{cases} \qquad \begin{cases} \frac{\partial N_1}{\partial t} = -\frac{1}{4}(1-s) = -\frac{1}{6} \\ \frac{\partial N_2}{\partial t} = -\frac{1}{4}(1+s) = -\frac{1}{3} \\ \frac{\partial N_3}{\partial t} = \frac{1}{4}(1+s) = \frac{1}{3} \\ \frac{\partial N_4}{\partial t} = \frac{1}{4}(1-s) = \frac{1}{6} \end{cases}$$

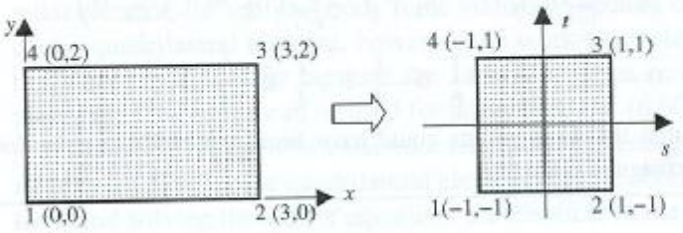


Figure 6.22 Mapping of a rectangular element

In addition, the Jacobian matrix can be calculated using Eq. (6.65), as

$$\begin{cases} \dfrac{\partial x}{\partial s} = -\dfrac{1}{4}\cdot 0 + \dfrac{1}{4}\cdot 3 + \dfrac{1}{4}\cdot 3 - \dfrac{1}{4}\cdot 0 = \dfrac{3}{2} \\ \dfrac{\partial y}{\partial s} = -\dfrac{1}{4}\cdot 0 + \dfrac{1}{4}\cdot 0 + \dfrac{1}{4}\cdot 2 - \dfrac{1}{4}\cdot 2 = 0 \\ \dfrac{\partial x}{\partial t} = -\dfrac{1}{6}\cdot 0 - \dfrac{1}{3}\cdot 3 + \dfrac{1}{3}\cdot 3 + \dfrac{1}{6}\cdot 0 = 0 \\ \dfrac{\partial y}{\partial t} = -\dfrac{1}{6}\cdot 0 - \dfrac{1}{3}\cdot 0 + \dfrac{1}{3}\cdot 2 + \dfrac{1}{6}\cdot 2 = 1 \end{cases}$$

$$[\mathbf{J}] = \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} \dfrac{3}{2} & 0 \\ 0 & 1 \end{bmatrix}, \quad [\mathbf{J}]^{-1} = \begin{bmatrix} \dfrac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

The Jacobian is positive, and the mapping is valid at this point. In fact, the Jacobian matrix is constant throughout the element. Note that the Jacobian matrix only has diagonal components, which means that the physical element is a rectangle. The horizontal dimension of the physical element is 1.5 times that of the reference element, and the vertical dimension is the same.

Using the inverse Jacobian matrix and the derivatives of the shape functions, we can calculate the following:

$$\begin{cases} \dfrac{\partial N_I}{\partial s} = \dfrac{\partial N_I}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial N_I}{\partial y}\dfrac{\partial y}{\partial s} \\ \dfrac{\partial N_I}{\partial t} = \dfrac{\partial N_I}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial N_I}{\partial y}\dfrac{\partial y}{\partial t} \end{cases} \Rightarrow \begin{Bmatrix} \dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix} = [\mathbf{J}] \begin{Bmatrix} \dfrac{\partial N_I}{\partial x} \\ \dfrac{\partial N_I}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial N_I}{\partial x} \\ \dfrac{\partial N_I}{\partial y} \end{Bmatrix} = [\mathbf{J}]^{-1} \begin{Bmatrix} \dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix} = \begin{bmatrix} \dfrac{2}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix} = \begin{Bmatrix} \dfrac{2}{3}\dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix}$$

Using the derivatives of the shape functions, the strains can be calculated using Eq. (6.68), as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} u_I = \sum_{I=1}^{4} \frac{2}{3}\frac{\partial N_I}{\partial s} u_I$$

$$= \frac{2}{3}\left(-\frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 2 - \frac{1}{4}\cdot 0\right) = \frac{1}{2}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} v_I = \sum_{I=1}^{4} \frac{\partial N_I}{\partial t} v_I$$

$$= -\frac{1}{6}\cdot 0 - \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 + \frac{1}{6}\cdot 2 = \frac{2}{3}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \sum_{I=1}^{4}\left(\frac{\partial N_I}{\partial y} u_I + \frac{\partial N_I}{\partial x} v_I\right) = \sum_{I=1}^{4}\left(\frac{\partial N_I}{\partial t} u_I + \frac{2}{3}\frac{\partial N_I}{\partial s} v_I\right)$$

$$= -\frac{1}{6}\cdot 0 - \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 2 + \frac{1}{6}\cdot 0 - \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1 - \frac{1}{4}\cdot 2 = \frac{1}{12}$$

The reader can verify that the same results could have been obtained using the formulas in Eq. (6.57) derived for rectangular elements.

6.6.4 Finite Element Matrix Equation

As in the case of the CST element, the element stiffness matrix can be calculated from the strain energy of the element. By substituting for strains from Eq. (6.68) into the strain energy in Eq. (6.9) we have

$$\begin{aligned} U^{(e)} &= \frac{h}{2}\iint_A \{\varepsilon\}^T[\mathbf{C}]\{\varepsilon\}\,dA^{(e)} \\ &= \frac{h}{2}\{\mathbf{q}^{(e)}\}^T \iint_A \underset{8\times 3}{[\mathbf{B}]^T}\underset{3\times 3}{[\mathbf{C}]}\underset{3\times 8}{[\mathbf{B}]}\,dA\{\mathbf{q}^{(e)}\} \\ &\equiv \frac{1}{2}\{\mathbf{q}^{(e)}\}^T \underset{8\times 8}{[\mathbf{k}^{(e)}]}\{\mathbf{q}^{(e)}\} \end{aligned} \tag{6.69}$$

where $[\mathbf{k}^{(e)}]$ is the element stiffness matrix. Calculation of the element stiffness matrix has two challenges. First, the integration domain is a general quadrilateral shape, and, second, the displacement-strain matrix $[\mathbf{B}]$ is written in (s, t) coordinates. Thus, the integration in Eq. (6.69) is not trivial. Using the idea of mapping the physical element into the reference element, we can perform the integration in Eq. (6.69) in the reference element. Since the reference element is a square and is defined in (s, t) coordinates, the above two challenges can be resolved simultaneously. Again, the Jacobian plays an important role in transforming the integral to the reference element. Let us consider an infinitesimal area dA of the physical element is mapped into an infinitesimal rectangle $ds{\cdot}dt$ in the reference element. Then, the relation between the two areas becomes

$$dA = |\mathbf{J}|dsdt \tag{6.70}$$

Thus, the element stiffness matrix in the reference element can be written as

$$[\mathbf{k}^{(e)}] = h\iint_A [\mathbf{B}]^T[\mathbf{C}][\mathbf{B}]\,dA \equiv h\int_{-1}^{1}\int_{-1}^{1} [\mathbf{B}]^T[\mathbf{C}][\mathbf{B}]|\mathbf{J}|\,dsdt \tag{6.71}$$

Although the integration has been transformed to the reference element, still the integration in Eq. (6.71) is not trivial because the integrand cannot be written down as an explicit function of s and t. Note that the matrix $[\mathbf{B}]$ includes the inverse of the Jacobian matrix. Thus, it is going to be extremely difficult, if not impossible, to integrate Eq. (6.71) analytically. However, since the integral domain is a square, numerical integration can be used to calculate the element stiffness matrix. Numerical integration methods using Gauss quadrature, which is the most popular method, will be discussed in the following section. Similar to the other elements, the strain energy of entire solid can be obtained using Eq. (6.29), which involves the assembly process.

The potentials of applied loads can be obtained by following a similar procedure as the CST and rectangular elements. The potential energy of concentrated forces and distributed forces will be the same as that of the CST element. The potential energy of the body force can be calculated using Eq. (6.60), except that the transformation in Eq. (6.70) should be used so that the integration be performed in the reference element. For rectangular element, the uniform body force yields the equally divided nodal forces. In the case of the quadrilateral element, however, the work-equivalent nodal forces will not divide the body force equally because the Jacobian is not constant within the element. The numerical integration can be used for integrating Eq. (6.60).

Using the principle of minimum total potential energy in Eq. (6.41), a similar global matrix equation for the quadrilateral elements can be obtained. Applying boundary conditions and solving the matrix equations are identical to the CST element. After solving for

Table 6.2 Gauss Quadrature Points and Weights

NG	Integration points (s_i)	Weights (w_i)	Exact polynomial degree
1	0.0	2.0	1
2	±0.5773502692	1.0	3
3	±0.7745966692	0.5555555556	5
	0.0	0.8888888889	
4	±0.8611363116	0.3478546451	7
	±0.3399810436	0.6521451549	
5	±0.9061798459	0.2369268851	9
	±0.5384693101	0.4786286705	
	0.0	0.5688888889	

nodal displacements, strain and stress in an element can be calculated using Eqs. (6.68) and (6.43), respectively.

6.7 NUMERICAL INTEGRATION

As discussed before, it is not trivial to analytically integrate the element stiffness matrix and body force for the quadrilateral element. Although there are many numerical integration methods available, Gauss quadrature is the preferred method in the finite element analysis because it requires fewer function evaluations compared to other methods. We will explain the one-dimensional Gauss quadrature first.

In the Gauss quadrature, the integrand is evaluated at predefined points (called Gauss points). The sum of these integrand values, multiplied by integration weights (called Gauss weight), provides an approximation to the integral:

$$I = \int_{-1}^{1} f(s)\,ds \approx \sum_{i=1}^{n} w_i f(s_i) \tag{6.72}$$

where n is the number of Gauss points, s_i the Gauss points, w_i the Gauss weights, and $f(s_i)$ the function value at the Gauss point s_i. The locations of Gauss points and weights are derived in such a way that with n points, a polynomial of degree $2n - 1$ can be integrated exactly. Note that the integral domain is normalized, i.e., $[-1, 1]$. The Gauss quadrature performs well when the integrand is a smooth function. Table 6.2 shows the locations of the Gauss points and corresponding weights.

EXAMPLE 6.9 ***Numerical Integration***

Evaluate the following integral using Gauss quadrature with 1~4 integration points. Compare the numerical integration results with the analytical integration.

$$I = \int_{-1}^{1} (8x^7 + 7x^6)\,dx$$

SOLUTION It can be easily verified that the exact integral will yield $I = 2$. Now we assume that the exact integral is unknown and calculate its approximate value using Gauss quadrature.

(a) 1-point integral:

$$s_1 = 0, \quad f(s_1) = 0, \quad w_1 = 2$$
$$I = w_1 f(s_1) = 2 \times 0 = 0$$

$2n-1 = 7 \rightarrow n = 4$

Obviously, the one-point integral is not accurate.

(b) 2-point integral:

$$s_1 = -.577, \quad f(s_1) = 8(-.577)^7 + 7(-.577)^6 = .0882 \quad w_1 = 1$$
$$s_2 = .577, \quad f(s_2) = 8(.577)^7 + 7(.577)^6 = .4303, \quad w_2 = 1$$
$$I = w_1 f(x_1) + w_2 f(x_2) = .0882 + .4303 = .5185$$

2-point integral still has a large error because it is accurate only up to the third-order polynomial.

(c) 3-point integral:

$$s_1 = -.7746, \quad f(s_1) = .17350, \quad w_1 = .5556$$
$$x_2 = 0.0, \quad f(s_2) = 0.0 \quad w_2 = .8889$$
$$x_3 = .7746, \quad f(s_3) = 2.8505, \quad w_3 = .5556$$
$$I = w_1 f(s_1) + w_2 f(s_2) + w_3 f(s_3) = .5556(.17350 + 2.8505) = 1.6800$$

(d) 4-point integral:

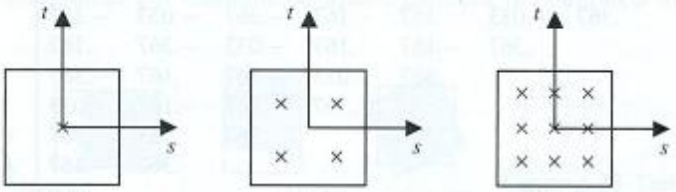
$$s_1 = -.8611, \quad f(s_1) = .0452, \quad w_1 = .3479$$
$$s_2 = -.3400, \quad f(s_2) = .0066, \quad w_2 = .6521$$
$$s_3 = .3400, \quad f(s_3) = .0150, \quad w_3 = .6521$$
$$s_3 = .8611, \quad f(s_3) = 5.6638, \quad w_3 = .3479$$
$$I = w_1 f(s_1) + w_2 f(s_2) + w_3 f(s_3) + w_4 f(s_4) = 2.0$$

Note that the 4-point integral is exact up to seventh-order polynomials. Since the given problem is seventh-order polynomial, the numerical integration is exact.

Two-dimensional Gauss integration formulas can be obtained by combining two one-dimensional Gauss quadrature formulas as shown below:

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds dt$$
$$\approx \int_{-1}^{1} \sum_{i=1}^{m} w_i f(s_i, t)\,dt \tag{6.73}$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{m} w_i w_j f(s_i, t_j)$$

where s_i and t_j are Gauss points, m is the number of Gauss points in s direction, n is the number of Gauss points in t direction, and w_i and w_j are Gauss weights. The total number of Gauss points becomes $m \times n$. Figure 6.23 shows few commonly used integration formulas.



(a) 1×1 (b) 2×2 (c) 3×3

Figure 6.23 Gauss integration points in two-dimensional parent elements

The element stiffness matrix in Eq. (6.71) can be evaluated using 2×2 Gauss integration formulas:

$$[\mathbf{k}^{(e)}] = h \int_{-1}^{1}\int_{-1}^{1} [\mathbf{B}]^T[\mathbf{C}][\mathbf{B}]|\mathbf{J}|\, dsdt \approx h\sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j [\mathbf{B}(s_i,t_j)]^T[\mathbf{C}][\mathbf{B}(s_i,t_j)]|\mathbf{J}(s_i,t_j)| \tag{6.74}$$

EXAMPLE 6.10 ***Numerical Integration of Element Stiffness Matrix***

Calculate the element stiffness matrix of the square element shown in Figure 6.24 using (a) 1×1 Gauss quadrature and (b) 2×2 Gauss quadrature. Compare the numerically integrated element stiffness matrix with the exact one calculated using Eq. (6.59). Assume plane stress with thickness $h = 0.1$ m, Young's modulus $E = 10$ GPa, and Poisson's ratio $\nu = 0.25$.

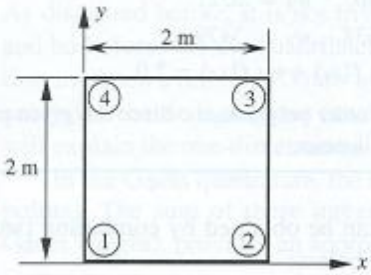


Figure 6.24 Numerical integration of a square element

SOLUTION Since the element size is the same as that of the reference element, the Jacobian matrix becomes the identity matrix. Thus, from Eq. (6.68), the displacement-strain matrix [**B**] becomes

$$[\mathbf{B}] = \frac{1}{4}\begin{bmatrix} -1+t & 0 & 1-t & 0 & 1+t & 0 & -1-t & 0 \\ 0 & -1+s & 0 & -1-s & 0 & 1+s & 0 & 1-s \\ -1+s & -1+t & -1-s & 1-t & 1+s & 1+t & 1-s & -1-t \end{bmatrix}$$

(a) 1×1 Gauss quadrature uses one-point integration at $(s, t) = (0, 0)$ with weight equal to 4. The [**B**] matrix at this point becomes

$$[\mathbf{B}] = \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

Then, the numerical integration of the element stiffness matrix becomes

$$[\mathbf{k}_1] \approx h w_1 w_1 [\mathbf{B}(0,0)]^T[\mathbf{C}][\mathbf{B}(0,0)]$$

$$= 10^9 \begin{bmatrix} .367 & .167 & -.167 & -.033 & -.367 & -.167 & .167 & .033 \\ & .367 & .033 & .167 & -.167 & -.367 & -.033 & -.167 \\ & & .367 & -.167 & .167 & -.033 & -.367 & .167 \\ & & & .367 & .033 & -.167 & .167 & -.367 \\ & & & & .367 & .167 & -.167 & -.033 \\ & & & & & .367 & .033 & .167 \\ & & & & & & .367 & -.167 \\ & & \text{Symmetric} & & & & & .367 \end{bmatrix}$$

(b) For 2×2 Gauss quadrature, we need four integration points and weights are unit.

Integration point	s	t
1	−.5773502692	−.5773502692
2	+.5773502692	−.5773502692
3	+.5773502692	+.5773502692
4	−.5773502692	+.5773502692

Then, the numerical integration of the element stiffness matrix in Eq. (6.74) becomes

$$[\mathbf{k}_2] \approx h \sum_{i=1}^{2} \sum_{j=1}^{2} w_i w_j [\mathbf{B}(s_i, t_j)]^T [\mathbf{C}][\mathbf{B}(s_i, t_j)] |\mathbf{J}(s_i, t_j)|$$

$$= 10^9 \begin{bmatrix} .489 & .167 & -.289 & -.033 & -.244 & -.167 & .044 & .033 \\ & .489 & .033 & .044 & -.167 & -.244 & -.033 & -.289 \\ & & .489 & -.167 & .044 & -.033 & -.244 & .167 \\ & & & .489 & .033 & -.289 & .167 & -.244 \\ & & & & .489 & .167 & -.289 & -.033 \\ & & & & & .489 & .033 & .044 \\ & & \text{Symmetric} & & & & .489 & -.167 \\ & & & & & & & .489 \end{bmatrix}$$

Using the exact stiffness in Eq. (6.59), we can find that the element stiffness matrix obtained from 2×2 Gauss quadrature is exact.

In general, the 2×2 Gauss quadrature is not exact for quadrilateral elements. The exact results in the above example occur because the element shape is a square.

6.7.1 Lower-Order Integration and Extra Zero-Energy Modes

It is important that the proper order of Gauss quadrature should be used. Otherwise, the element may show undesirable behavior. One of the well-known phenomena of lower-order integration is *extra zero energy modes*. The zero-energy mode is the deformation of an element without changing its strain energy. In plane solids, there are three types of deformations (more precisely, motions) that do not change the strain energy: x-translation, y-translation, and z-rotation. Figure 6.25 illustrates these modes. Since the relative locations of nodes do not change, the stress and strain of the elements are zero, and the strain energy remains constant. In finite element analysis, these modes should be fixed by applying displacement boundary conditions. Otherwise, the stiffness matrix will be singular and there will be no unique solution.

While the zero-energy modes in Figure 6.25 are proper modes, there are improper modes, called extra zero-energy modes, which often occurs when an element is under-integrated. For example, a square element is integrated using 1×1 Gauss quadrature

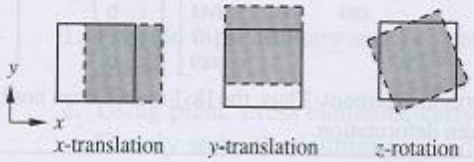


Figure 6.25 Three rigid-body modes of plane solids

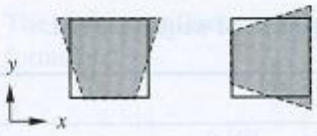


Figure 6.26 Two extra zero-energy modes of plane solids

there will be two extra zero-energy modes in addition to the three rigid-body modes. Figure 6.26 illustrates the two extra zero-energy modes of plane solids. It is clear that the element is being deformed but the centroid (the quadrature point) of the element does not experience any deformation and hence the strain energy remains constant. In other words, the element will deform without having externally applied forces, which is a numerical artifact. Thus, the extra zero-energy modes must be removed in order to obtain meaningful deformation.

The most common way of checking the extra zero-energy modes is using eigen values of the stiffness matrix. For a plane solid, the number of zero eigen values must be equal to three. However, the element stiffness matrix with 1×1 integration will have five zero eigen values corresponding to five zero energy modes shown in Figure 6.25 and Figure 6.26. In the following example, we will show another method of checking the extra zero-energy modes.

EXAMPLE 6.11 *Extra Zero-Energy Modes*

Consider two stiffness matrices of the square element in Example 6.10: $[\mathbf{k}_1]$ for 1×1 integration and $[\mathbf{k}_2]$ for 2×2 integration. When nodal displacements are given as $\{\mathbf{q}\}^T = \{0.1, 0, -0.1, 0, 0.1, 0, -0.1, 0\}$, check the reaction forces and determine if the stiffness matrix has extra zero-energy mode.

SOLUTION

(a) For $[\mathbf{k}_1]$ (1×1 integration), the reaction force can be calculated by multiplying the stiffness matrix with the nodal displacements as

$$[\mathbf{k}_1]\{\mathbf{q}\} = 10^9 \begin{bmatrix} .367 & .167 & -.167 & -.033 & -.367 & -.167 & .167 & .033 \\ & .367 & .033 & .167 & -.167 & -.367 & -.033 & -.167 \\ & & .367 & -.167 & .167 & -.033 & -.367 & .167 \\ & & & .367 & .033 & -.167 & .167 & -.367 \\ & & & & .367 & .167 & -.167 & -.033 \\ & & & & & .367 & .033 & .167 \\ & & \text{Symmetric} & & & & .367 & -.167 \\ & & & & & & & .367 \end{bmatrix} \begin{Bmatrix} 0.1 \\ 0.0 \\ -0.1 \\ 0.0 \\ 0.1 \\ 0.0 \\ -0.1 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

No force is required to deform the element. Thus, the $[\mathbf{k}_1]$ matrix has extra zero-energy mode.

(b) For $[\mathbf{k}_2]$ (2×2 integration), the reaction force can be calculated by multiplying the stiffness matrix with the nodal displacements as

$$[\mathbf{k}_2]\{\mathbf{q}\} = 10^9 \begin{bmatrix} .489 & .167 & -.289 & -.033 & -.244 & -.167 & .044 & .033 \\ & .489 & .033 & .044 & -.167 & -.244 & -.033 & -.289 \\ & & .489 & -.167 & .044 & -.033 & -.244 & .167 \\ & & & .489 & .033 & -.289 & .167 & -.244 \\ & & & & .489 & .167 & -.289 & -.033 \\ & & & & & .489 & .033 & .044 \\ & & \text{Symmetric} & & & & .489 & -.167 \\ & & & & & & & .489 \end{bmatrix} \begin{Bmatrix} 0.1 \\ 0 \\ -0.1 \\ 0 \\ 0.1 \\ 0 \\ -0.1 \\ 0 \end{Bmatrix} = 10^7 \begin{Bmatrix} 4.89 \\ 0 \\ -4.89 \\ 0 \\ 4.89 \\ 0 \\ -4.89 \\ 0 \end{Bmatrix}$$

Non-zero nodal forces are required to deform the element. Thus, the $[\mathbf{k}_2]$ matrix does not have extra zero-energy mode corresponding to the given deformation.

6.8 PROJECT

Project 6.1 – Accuracy and Convergence Analysis of a Cantilever Beam

In this project, we want to compare the finite element results of plane solid elements with those of uniaxial bar and beam elements. Consider a cantilever beam shown in Figure 6.27 under horizontal and transverse forces at the tip. The beam has a square cross-section of 0.1 m × 0.1 m, length of $L = 1$ m, Young's modulus $E = 207$ GPa, and Poisson's ratio $\nu = 0.3$.

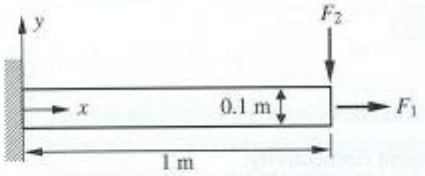


Figure 6.27 Cantilever beam model

Part I

(a) Consider the case of $F_1 = 100$ N and $F_2 = 0$. Solve the problem using a uniaxial bar element to find the elongation $u(x)$. Calculate ε_{xx} and σ_{xx}. Assume that $\sigma_{yy} = \sigma_{zz} = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0$. Compare the results with analytical solution.

(b) Consider the case of $F_1 = 0$ and $F_2 = 500$ N. Solve the problem using a beam element to find the deflection $w(x)$. Calculate ε_{xx} and σ_{xx}. Assume that $\sigma_{yy} = \sigma_{zz} = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0$. Compare the results with analytical solution. Plot σ_{xx} as a function of y at $x = L/2$.

Part II

(a) Consider the case of $F_1 = 100$ N and $F_2 = 0$. Solve the problem using: (i) 20 CST elements and (ii) 10 rectangular elements to find the elongation $u(x)$. Calculate ε_{xx} and σ_{xx}. Compare the results with those from Part I. Explain the results using interpolation scheme.

(b) Consider the case of $F_1 = 0$ and $F_2 = 500$ N. Solve the problem using: (i) 20 CST elements and (ii) 10 rectangular elements to find the deflection $w(x)$. Calculate ε_{xx} and σ_{xx}. Compare the results with those of Part I. Explain the results using interpolation scheme.

(c) Consider the case of $F_1 = 0$ and $F_2 = 500$ N. Perform convergence study by gradually decreasing element size and show the deflection and stress converge to the exact solution.

Project 6.2 – Design of a Torque-arm

A torque arm shown in Figure 6.28 is under horizontal and vertical loads transmitted from a shaft at the right hole, while the left hole is fixed. Assume: Young's modulus = 206.8 GPa, Poisson's ratio = 0.29, and thickness = 1.0 cm.

1. Provide a preliminary analysis result that can estimate the maximum von Mises stress.
2. Using plane stress elements, carry out finite element analysis for the given loads. Clearly state all assumptions and simplifications that you adopted in modeling.

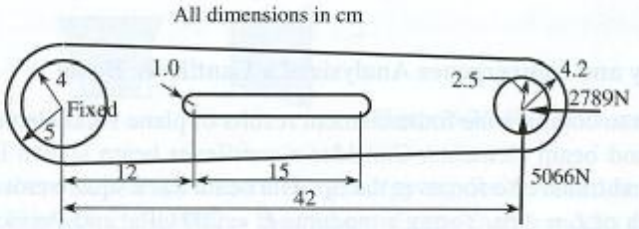


Figure 6.28 Dimensions of torque arm model

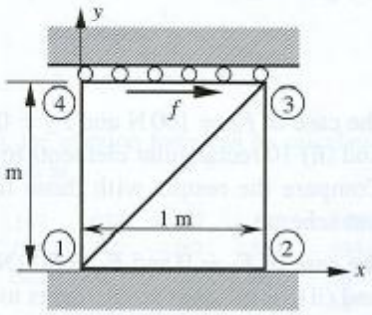
Carry out convergence study and determine the size of elements for a reasonably accurate solution.

6.9 EXERCISE

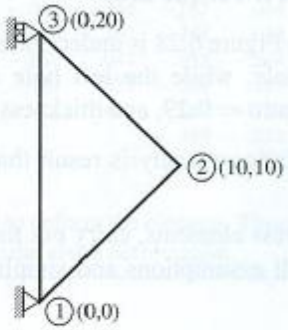
1. Repeat Example 6.2 with the following element connectivity:

   Element 1: 1–2–4
   Element 2: 2–3–4

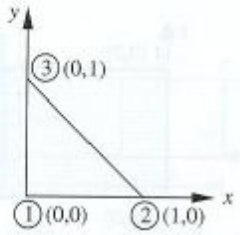
   Does the different element connectivity change the results?
2. Solve Example 6.2 using one of the finite element programs given in the Appendix.
3. Using two CST elements, solve the simple shear problem depicted in the figure and determine whether the CST elements can represent the simple shear condition accurately. Material properties are given as $E = 10\,\text{GPa}$, $\nu = 0.25$, and thickness is $h = 0.1\,\text{m}$. The distributed force $f = 100\,\text{kN/m}^2$ is applied at the top edge.



4. Solve Example 6.4 using one of finite element programs in the Appendix.
5. A structure shown in the figure is modeled using one triangular element. Plane strain assumption is used.
   (a) Calculate the strain-displacement matrix [**B**].
   (b) When nodal displacements are given by $\{u_1, v_1, u_2, v_2, u_3, v_3\} = \{0, 0, 2, 0, 0, 1\}$, calculate element strains.



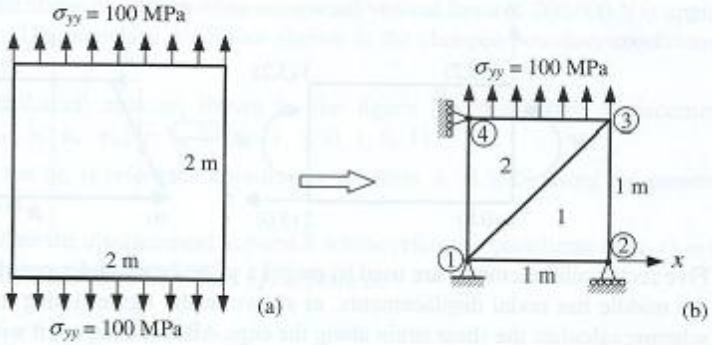
6. Calculate the shape function matrix [**N**] and strain-displacement matrix [**B**] of the triangular element shown in the figure.



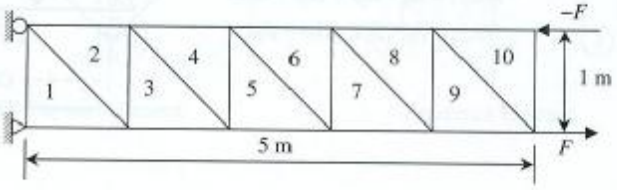
7. The coordinate of the nodes and corresponding displacements in a triangular element are given in the table. Calculate the displacement u and v and strains ε_{xx}, ε_{yy}, and γ_{xy} at the centroid of the element given by the coordinates (1/3, 1/3)

Node	x (m)	y (m)	u (m)	v (m)
1	0	0	0	0
2	1	0	0.1	0.2
3	0	1	0	0.1

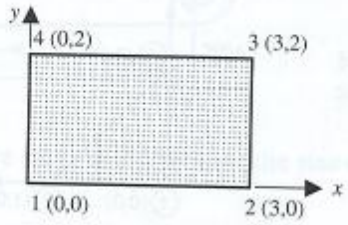
8. A 2 m × 2 m × 1 mm square plate with $E = 70$ GPa and $\nu = 0.3$ is subjected to a uniformly distributed load as shown in Figure (a). Due to symmetry, it is sufficient to model one quarter of the plate with artificial boundary conditions, as shown in Figure (b). Use two triangular elements to find the displacements, strains, and stresses in the plate. Check the answers using simple calculations from mechanics of materials.



9. A beam problem under the pure bending moment is solved using CST finite elements, as shown in the figure. Assume $E = 200$ GPa and $\nu = 0.3$. The thickness of the beam is 0.01 m. To simulate the pure bending moment, two opposite forces $F = \pm 100,000$ N are applied at the end of the beam. Using any available finite element program, calculate the stresses in the beam along the neutral axis and top and bottom surfaces. Compare the numerical results with the elementary theory of beam. Provide an element stress contour plot for σ_{xx}.



10. For a rectangular element shown in the figure, displacements at four nodes are given by $\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{0.0, 0.0, 1.0, 0.0, 2.0, 1.0, 0.0, 2.0\}$. Calculate displacement (u, v) and strain ε_{xx} at point $(x, y) = (2, 1)$.



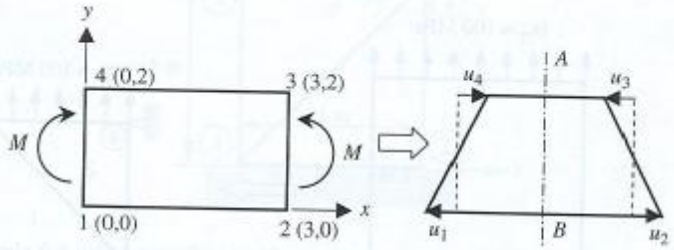
11. Six rectangular elements are used to model the cantilevered beam shown in the figure. Sketch the graph of σ_{xx} along the top surface that a finite element analysis would yield. There is no need to actually solve the problem, but use your knowledge of shape functions for rectangular elements.

12. A rectangular element as shown in the figure is used to represent a pure bending problem. Due to the bending moment M, the element is deformed as shown in the figure with displacement $\{\mathbf{q}\} = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}^T = \{-1, 0, 1, 0, -1, 0, 1, 0\}^T$.

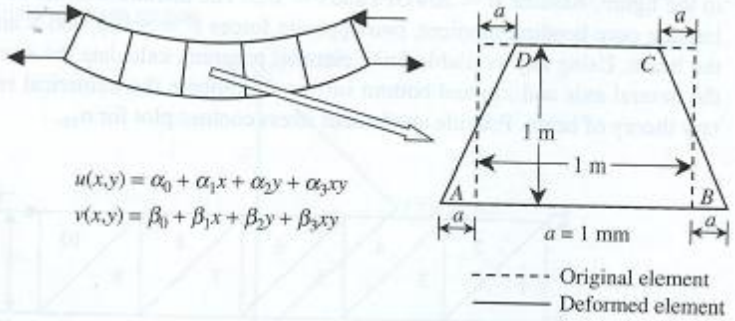
(a) Write the mathematical expressions of strain component $\varepsilon_{xx}, \varepsilon_{yy}$, and γ_{xy}, as functions of x and y.

(b) Does the element satisfy pure bending condition? Explain your answer.

(c) If two CST elements are used by connecting nodes 1–2–4 and 4–2–3, what will be ε_{xx} along line A-B?

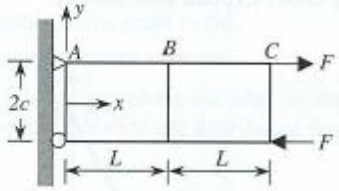


13. Five rectangular elements are used to model a plane beam under pure bending. The element in the middle has nodal displacements, as shown in the figure. Using the bilinear interpolation scheme, calculate the shear strain along the edge AB and compare it with the exact shear strain.



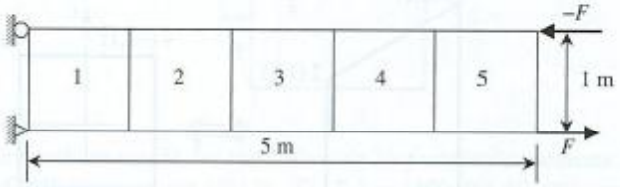
$$u(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy$$
$$v(x,y) = \beta_0 + \beta_1 x + \beta_2 y + \beta_3 xy$$

14. A uniform beam is modeled by two rectangular elements with thickness b. Qualitatively, and without performing calculations, plot σ_{xx} and τ_{xy} along the top edge from A to C, as predicted by FEA. Also, plot the exact stresses according to beam theory.



15. A beam problem under the pure bending moment is solved using five rectangular finite elements, as shown in the figure. Assume $E = 200$ GPa and $\nu = 0.3$ are used. The thickness of the beam is 0.01 m. To simulate the pure bending moment, two opposite forces $F = \pm 100,000$ N are applied at the end of the beam. Using a commercial FE program, calculate strains in the beam along the bottom surface. Draw graphs of ε_{xx} and γ_{xy}, with x-axis being the beam length. Compare the numerical results with the elementary theory of beam. Provide an explanation for the differences, if any. Is the rectangular element stiff or soft compared to the CST element?

Normally, a commercial finite element program provides stress and strain at the corners of the element by averaging with stresses at the adjacent elements. Thus, you may use nodal displacement data from FE code to calculate strains along the bottom surface of the element. Calculate the strains at about 10 points in each element for plotting purpose. Make sure that the commercial program uses the standard Lagrange shape function.



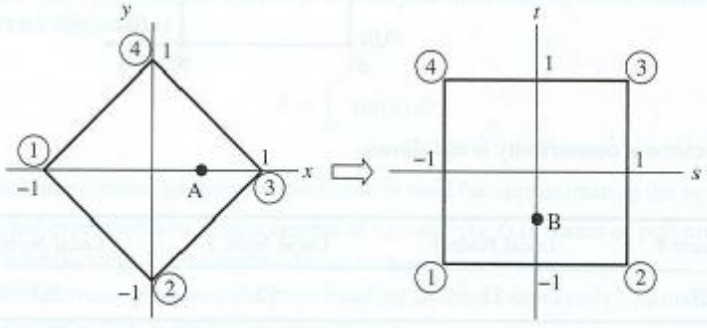
Repeat the above procedure when an upward vertical force of 200,000 N is applied at the tip of the beam. Use boundary conditions similar to the clamped boundary conditions of a cantilevered beam.

16. The quadrilateral element shown in the figure has the nodal displacement of $\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{-1, 0, -1, 0, 0, 1, 0, 1\}$.

(a) Find the (s, t) reference coordinates of point A (0.5, 0) using iso-parametric mapping method.

(b) Calculate the displacement at point B whose reference coordinate is $(s, t) = (0, -0.5)$

(c) Calculate the Jacobian matrix [J] at point B.



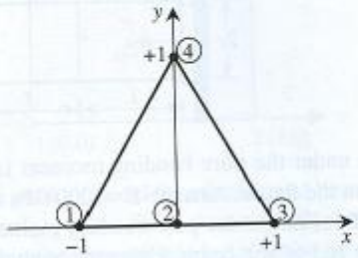
Physical Element Reference Element

17. A four-node quadrilateral element is defined as shown in the figure.

(a) Find the coordinates in the reference element corresponding to $(x, y) = (0, 0.5)$.

(b) Calculate the Jacobian matrix as a function of s and t.

(c) Is the mapping valid? Explain your answer.

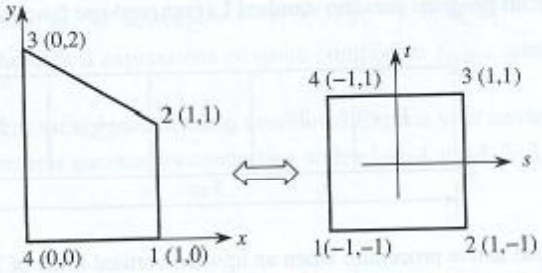


18. A quadrilateral element in the figure is mapped into the reference element.

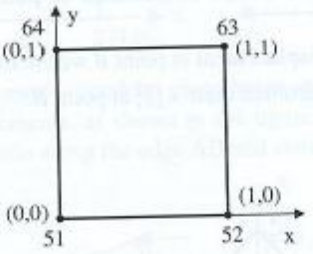
(a) A point P has a coordinate $(x, y) = (½, y)$ in the physical element and $(s, t) = (-½, t)$ in the parent element. Find the y and t coordinates of the point using iso-parametric mapping.

(b) Calculate the Jacobian matrix at the center of the element.

(c) Is the mapping valid? Explain your answer.



19. Consider the plane stress four-node element shown below. Its global node numbers are shown in the figure. The coordinates of the nodes in the global x-y coordinate systems is shown next to each node.



The element connectivity is as follows:

Element #	Local Node 1	Local Node 2	Local Node 3	Local Node 4
27	51	52	63	64

Nodal displacement vector $= \{\mathbf{q}\}^T = \{u_{51}, v_{51}, u_{52}, v_{52}, u_{63}, v_{63}, u_{64}, v_{64}\} = \{0, 0, 0.1, 0, 0.1, 0.1, 0, 0\}$.

(a) Determine the displacement at the point $(x, y) = (0.75, 0.75)$ by interpolating the nodal displacements.

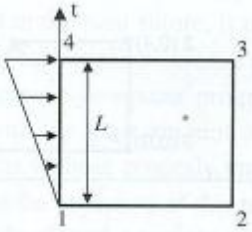
(b) Compute the Jacobian matrix at the point in (b).

(c) Compute strain ε_{yy} at the center of the element.

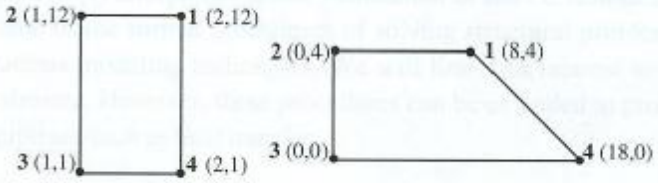
20. A linearly varying pressure p is applied along the edge of the four-node element shown in the figure. The finite element method converts the distributed force into an equivalent set of nodal forces $\{\mathbf{F}^e\}$ such that

$$\int_S \mathbf{u}^T \mathbf{T} dS = \{\mathbf{q}^{(e)}\}^T \{\mathbf{F}^{(e)}\}$$

where $\mathbf{T}$ is the applied traction (force per unit area) and $\mathbf{u}$ is the vector of displacements. Since the applied pressure is normal to the surface (in the x-direction), the traction can be expressed as $\mathbf{T} = \{p, 0\}^T$ where p can be expressed as $p = p_0(t+1)/2$, $t = -1$ at Node 1 and $t = +1$ at Node 4. The length of the edge is L. Integrate the left-hand side of the above equation to compute the work-equivalent nodal forces $\{\mathbf{F}^{(e)}\}$ when $\{\mathbf{q}^{(e)}\}^T = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}$.



21. Determine the Jacobian matrix for the following isoparametric elements. If the temperature at the nodes of both elements are $\{T_1, T_2, T_3, T_4\} = \{100, 90, 80, 90\}$, compute the temperature at the midpoint of the element and at the midpoint of the edge between connecting Nodes 1 and 4.



22. Integrate the following function using one-point and two-point numerical integration (Gauss quadrature). The exact integral is equal to 2. Compare the accuracy of the numerical integration with the exact solution.

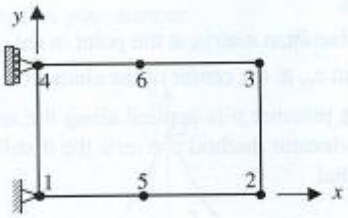
$$I = \int_0^{\pi} \sin(x)\, dx$$

23. A six-node finite element as shown in the figure is used for approximating the beam problem.

(a) Write the expressions of displacements $u(x,y)$ and $v(x,y)$ in terms of polynomials with unknown coefficients. For example, $u(x, y) = a_0 + a_1 x + \cdots$.

(b) Can this element represent the pure bending problem accurately? Explain your answer. Bending moment M is applied at the edge 2-3.

(c) Can this element represent a uniformly distributed load problem accurately? The distributed load q is applied along the edge 4-6-3.



24. Consider a quadrilateral element shown in the figure below. The nodal temperatures of the element are given as $\{T_1, T_2, T_3, T_4\} = \{80, 40, 40, 80\}$.

(a) Compute the expression of the temperature T along the line ξ that connects Nodes 3 and 1. For example, $T = 3 + 5\xi + 3\xi^2 + \cdots$. You can assume that $\xi = 0$ at Node 3 and $\xi = 1$ at Node 1. Plot the graph of $T(\xi)$ with respect to ξ.

(b) Compute the temperature gradient $\partial T/\partial x$ at the center of the element.