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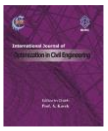

NODAL ORDERING OF GRAPHS FOR WAVEFRONT OPTIMIZATION USING NEURAL NETWORK AND WATER STRIDER ALGORITHMS

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ABSTRACT

In this paper, a neural network is trained for optimal nodal ordering of graphs to obtain a small wavefront using soft computing. A preference function consists of six inputs that can be seen as a generalization of Sloan's function. These six inputs represent the different connection characteristics of graph models. This research is done with the aim of comparing Sloan's theoretical numbering method with Sloan's developed method with neural networks and WSA meta-heuristic algorithm. Unlike the Sloan algorithm, which uses two fixed coefficients, six coefficients are used here, based on the evaluation of artificial neural networks. The weight of networks is obtained using Water Strider algorithm. Examples are included to demonstrate the performance of the present hybrid method.

Keywords: Graph ordering, wavefront, neural network, Metaheuristic algorithm, WSA.

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1. INTRODUCTION

Optimal analysis is defined as the analysis where the structural matrices (stiffness or flexibility) are sparse, well structured and well conditioned [1,2]. In order to have well structured matrices the nodes/elements should be ordered depending on the method to be used for the solution of the involved equations. For Gaussian elimination banded form is the most traditional form [3,4,5].

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For skyline solution the profile should be reduced [6,7,8] and for frontal solution frontwidth must be reduced [9,10,11,12,13].

Frontal method is particularly popular due to its capability to eliminate variables during the matrix assembly process [14]. In contrast, band and profile methods usually first explicitly construct the coefficient matrix and then perform the solution operations based on it.

These approaches, along with other solution techniques, can help reduce computational time and improve efficiency in finite element analyses [15]. In finite element analysis, in the case of one degree of freedom per node, performing a nodal ordering is equivalent to reordering the equations. In a more general problem with m degrees of freedom at each node, m coupled equations are generated for each node. In this case, reordering is usually performed on the node numbering of the graph model to reduce the bandwidth, profile, or wavefront, since the size of the problem is m-fold smaller than that.

Various algorithms have been developed for numbering degrees of freedom and for wavefront reduction in matrix systems. These algorithms aim to minimize the size of the wavefront and optimize the process of solving equations [16-19].

This research has been performed with the aim of comparing Sloan's theoretical numbering method with Sloan's developed method with neural networks and meta-heuristic WSA algorithm. A total of 425 data sets were collected from the results of numbering different diagrams with Sloan's algorithm. Node, ds, de, deg, nact and npre are the input parameters and the order of the nodes of the graph is estimated as the output variable. Unlike the Sloan algorithm which uses two constant coefficients, here, based on the evaluation of the artificial neural network, six coefficients are used. The weight of networks is obtained using Water Strider algorithm. Examples are included to demonstrate the performance of the present hybrid method.

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2. DEFINITIONS

A graph consists of nodes and members, the members are connected to each other with nodes, and there are members on both sides of the node. Two nodes of a graph are called adjacent if these nodes are end nodes of a member. A member is called incident with a node if the node is an end node of the member. The degree of a node is the number of members that interact with that node.

A tree is a subset of a graph in which there are no cycles. A cycle means a closed path in a graph. A spanning tree refers to a tree that contains all the nodes of a graph. The shortest root tree (SRT) is a tree whose distance from each node to the root is the shortest possible [20].

A contour of a 〖SRT〗 consists of all nodes with equal distance k from the root. The number of lines "d" of an SRT is known as its depth and the maximum number of nodes in a contour "w" defines the width of the SRT. Consider the solution of diffuse linear equations:

$$Dx = z \tag{1}$$

where the $n \times n$ matrix **D** is a sum of elemental matrices

$$D = \sum_{l=1}^{m} D^{[l]} \tag{2}$$

and the right-hand side vector **z** is of the form,

$$z = \sum_{l=1}^{m} z^{[l]} \tag{3}$$

In Equation (2) each matrix $\boldsymbol{D}^{[l]}$ has entries only in the principle submatrix corresponding to the variables in element l and represents contributions from this element. This principle submatrix is assumed to be dense. The matrix **D** may be non-symmetric but form (2) implies that it has a symmetric pattern. With reference to Equation (1), column *j* is said to be *active* at stage i , if $j \geq i$ and there is a non-zero entry in column *j* with a row index, *k*, such that $k \leq i$. Letting f_i denote the number of columns that are active at stage i, the *maximum frontwidth* of **D** is given by

$$F_{max} = max\{f_i\}, \;\; 1 \leq i \leq n \tag{4}$$

The root-mean-squared-error (RMSE) wavefront is defined to be

$$F_{rmse} = (\frac{1}{n}\sum_{i=1}^{n} f_i^2)^{1/2} \tag{5}$$

Assuming that n and the average value of f_i are reasonably large,it can be shown that a complete profile or front factorization approximately O(nF_{rmse}^2) operations.

3. SLOAN'S ALGORITHM FOR ELEMENT ORDERING

Sloan's algorithm consists of two steps, the first is the selection of pseudo-peripheral nodes and the second is numbering. The two steps are described below.

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3.1 Selection of pseudo-peripheral nodes

A pair of starting nodes is determined by the following method. This approach has been used by Duff et al. [21] which in turn is a modified version of George and Liu's previous algorithm [22].

Algorithm:

Step 1: Pick a node with minimum degree and call it v.

Step 2: Form an SRT whose root is v and decompose the subgraph containing the nodes into its subcontours.

Step 3: Select one node of minimum degree from each subcontours. Generate an SRT from each of such nodes, and choose the one corresponding to the smallest width.

Step 4: Repeat the process of Steps 2 and 3 as far as reduction in width of the SRT can be observed.

the root '*s*' and the selected end node '*e*' are the pseudo-peripheral nodes required.

3.2 Node reordering

In the node reordering step of Sloan's algorithm, initially, nodes are arbitrarily labeled with integers from 1 to N, where N is the number of nodes in the graph. New node labels are generated in one step. Each node to which a new label is assigned is called a post-active state. Any node that is adjacent to a post-active node, but does not have a post-active state, is defined as an active state. Nodes that are adjacent to an active node, but do not have an active or post-active state, have a pre-active state. Nodes that do not have post-active, active, or pre-active status are inactive [8].The following steps are taken:

Step 1: According to the given description, group the nodes as shown in Figure 1.

Step 2: Label the active and pre-active nodes for the next step.

Step 3: Get the priority for all candidate nodes. The priority calculation for node i is as follows:

$$N(i) = -W_1 \times r(i) - W_2 \times d(e,i) \tag{6}$$

where W_1 and W_2 are integer coefficients (set as $W_1 = 1$ and $W_2 = 2$ in the original Sloan's algorithm [14]); $d(e,i)$ is the distance of node i to the end node e and

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$$r(i) = h_i - b_i + t_i \quad (7)$$

where h_i is the degree of node i, b_i is the number of active and post-active nodes adjacent to node i, and t_i is zero if the node i is active and post-active, and unity otherwise. Here, $r(i)$ represents the increment of the frontwidth for labelling of the node i.

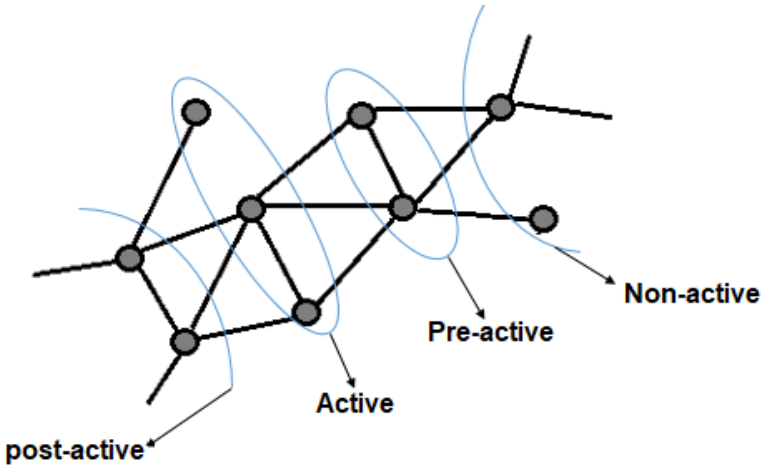

Figure 1: Different statues of node in the proses of labelling

Step 4: Select the node with higher priority and label it.

Step 5: Repeat Steps 1–4 until all the nodes are labeled [17].

King's algorithm is similar in nature to Sloan's algorithm, with the difference that, firstly, it does not need to select the start and end nodes, and secondly, in equation 6, W1 = 1 and W2 = 0. Figure 2 shows a sample graph with simple numbering and numbering with the Sloan method. As explained in section 3.1, by forming the SRT, we obtain the values of de (i) and ds(i) and the degree of the nodes, deg(i).

As an example, consider the graph shown in Figure 2(a). Sloan's algorithm leads to the numbering as shown in Figure 2(b). For this graph s = 1 and e = 20 are selected, and for a typical node i = 9, the above defined values are as ds(9) = 3, de(9) = 4 and deg(9) = 4. Similar to Sloan's method.

The number of nodes adjacent to the node i in the status of active is shown by nact(i), and the number of nodes adjacent to i in the status of pre-active is designated by npre(i).

For the graph of Figure 2(b), after labelling the nodes 1 and 2, these nodes will be in the state of post-active. The nodes 3 and 4 have active; the nodes 5 and 9 are in the state of pre-active; the rest of the nodes have non-active status. In this example, for the node 9, nact(9) = 2 and npre(9) = 2.

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Considering the above definitions, for each node i, the neural network has six input neurons. Selecting the starting node as assigning the label 1 to this node, the numbering starts. At each consequent step, after updating the status of the nodes, the input is given to the net and the output N (i) as the priority of the node i is obtained for all the candidate nodes.

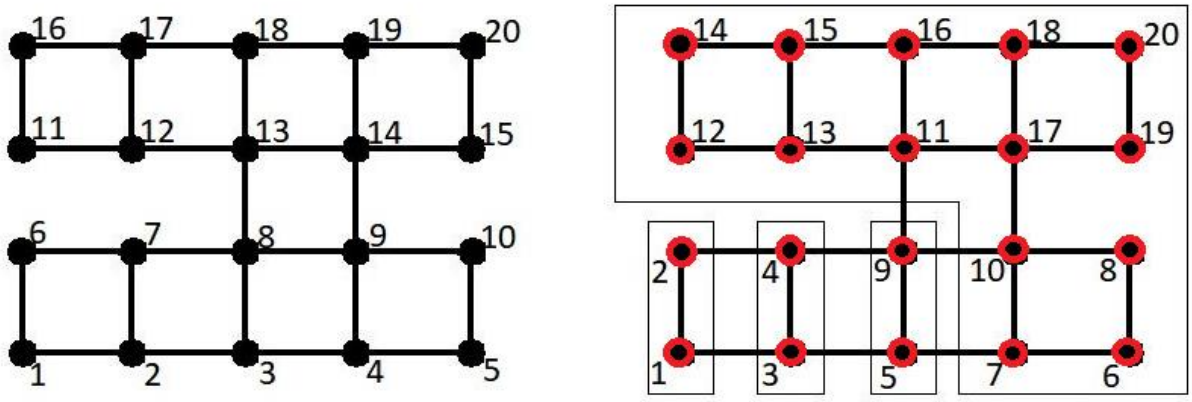

Figure 2: A simple example

4. HYBRID NEURAL NETWORK-WATER STRIDER ALGORITHM

In this paper a neural network is trained for optimal nodal ordering of graphs. In this approach ideas from neural networks and priority function are combined for evaluating suitable coefficients for the vectors of the priority function. The WSA is used for evaluation of the weights of the networks [23].

In Sloan's method, the priority function is a linear function of two graph parameters as in Equation (6), and the coefficients specifies the importance of each parameter. Following experimentations, Sloan's recommends the pair (W1 = 1 and W2 = 2) for the coefficients. However, the results of other researcher [10] indicate that, for some problems there are considerable advantages in using other values.

MLP consists of two types of elements: neurons (nodes) and weights (connections). Neurons are organized in layers: the first layer is called the input layer, which receives data from the external environment [24]. The last layer called the output layer that provides the output or responses from the network. Between the input and output layers, there is at least one hidden layer. The weights represent the strength of the connections, which determine the distribution of the data flow among the layers. Between any two neighboring layers, neurons are fully connected. For two layers with m and n neurons, there are m × n connections. Therefore, the weight between two adjacent layers can be represented as an m × n matrix.

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In this article, according to the number of databases used for training, the network shown in Figure 3 has two hidden layers, with 15 and 13 neurons. In this research, the Water Strider algorithm was used instead of the backpropagation method to obtain the weights in the neural network. 70% of data is used for training, 15% of data for testing and 15% of them for validation. Tan-sigmoid activation function as it is shown in Eq. (8), is used:

$$Tangent\ Sigmoid = \frac{(e^{x} - e^{-x})}{(e^{x} + e^{-x})} \tag{8}$$

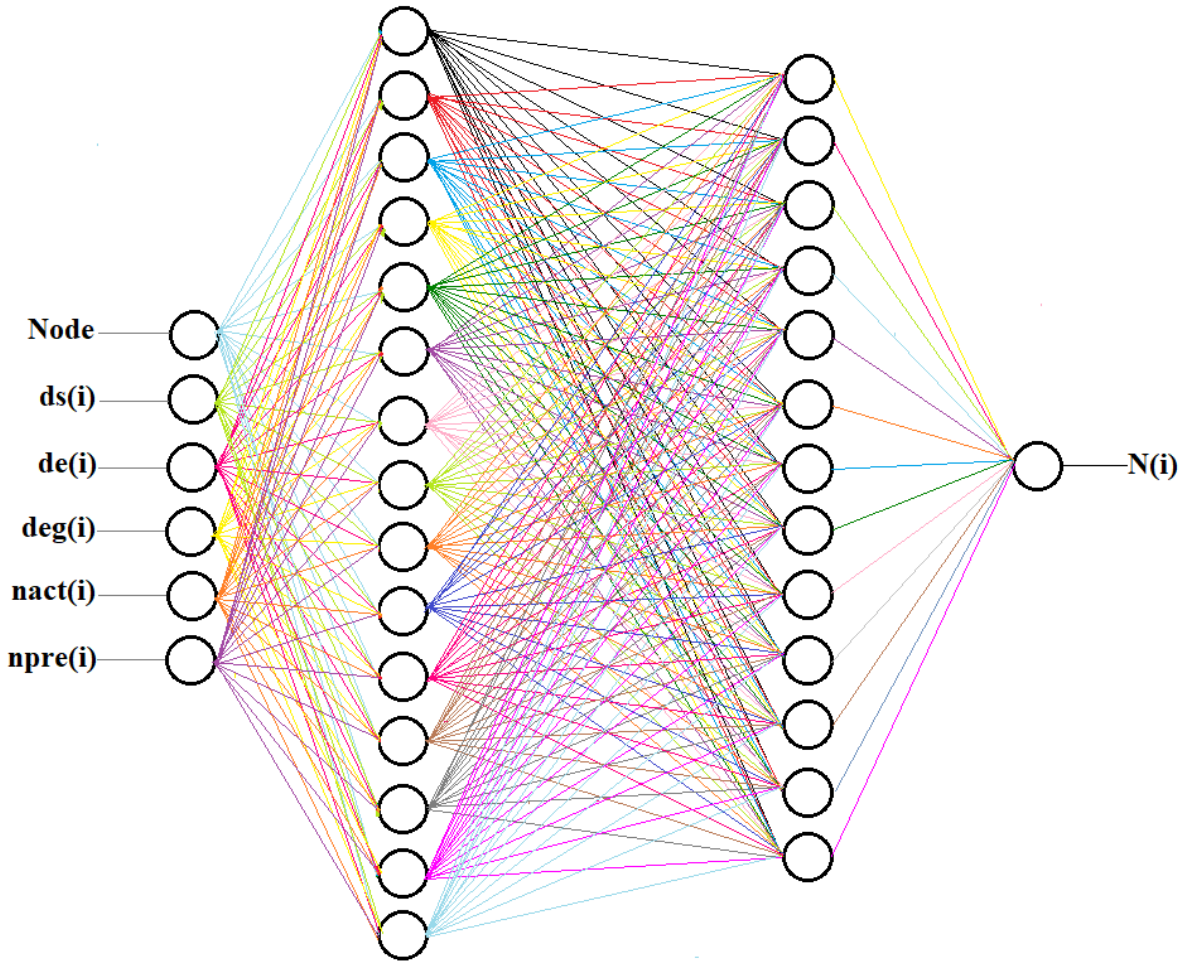

Figure 3: Topology of the neural network for the example

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5. CONSTRUCTION OF THE DATASET

For a collection of examples used for training, ordering the nodes can be performed using any available technique. One can for example use the Sloan's method. The graphs used for ordering, can be a single one or a group of graphs with similar topological properties. The dataset used in the study contained 425 data points obtained from different graphs with Sloan algorithm. Six parameters were used as input parameters, including, node, ds, de, deg, nact, and npre. The node number (N) is estimated as the output parameter. The input and target parameters' initial statistics are described in Table 1.

Table 1: The statistical description of the datasets

Parameters	Explanation	Symbols
X1	Number node of graph	node
X2	Distance of start node	ds
X3	Distance of end node	de
X4	Degree of node	deg
X5	The number of nodes adjacent to the node *i* in the status of active	nact
X6	The number of nodes adjacent to *i* in the status of pre-active	npre
Output	Node number	N

6. WATER STRIDER ALGORITHM

To describe the WSA algorithm, five main stages including birth, territory establishment, mating, feeding and death are modeled mathematically. Throughout the algorithm, the search space is defined as a lake containing different solution regions, and food serves as a metaphor for the objective function [22].

6.1. Initial birth

Initially, the water striders (WSs) are born with eggs distributed on the lake. A random distribution is assumed as Eq. (9)

$$WS_i^0 = Ub + rand(Ub - Lb), i = 1,2, \dots, nws \tag{9}$$

where WS_i^0 determines the initial position of ith water strider. Ub and Lb denote the upper and lower bounds corresponding to maximum and minimum allowable values, respectively; rand is a random number between 0 and 1; nws is the number of WSs. The initialized WSs are evaluated using an objective function to calculate the fitness of their position on the lake.

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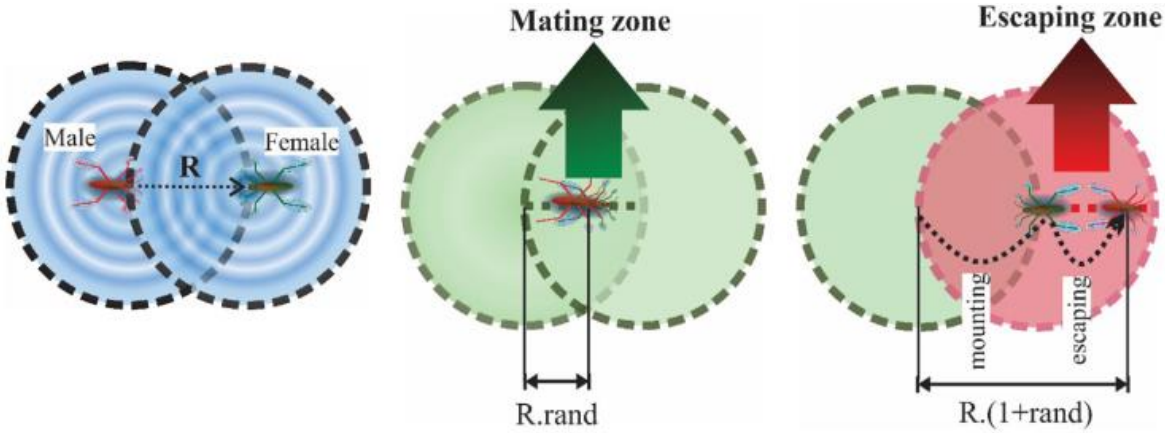
6.2. Territory establishment

WSs maintain territories to live, mate, and feed. To establish nt number of territories, the following technique is utilized for assigning the WSs to the territories. Firstly, WSs are sorted based on their fitness and are divided into $nws/\,nt$ groups orderly. The j^{th} member of each group is assigned to j^{th} territory, where$j\ =\ 1.2.,,,.nt$. Hence $nws/\,nt$ number of WSs live inside each territory. Since the females usually find the best place of each territory for feeding, the positions in each territory with the best and worst fitness are considered as the female (optimal foraging-habitat) and male (keystone) positions, respectively.



(a) The signal exchange between male and female WSs for mating

(b)successful mating of WSs after attraction

(c)Unsuccessful mating of WSs consisting of Mating and escaping levels

Figure 4: The mating behavior of water striders and position updating

6.3. Mating

Mating is a remarkable process in the life of water striders. As aforementioned, the keystone sends courtship calling ripples and the female respond by sending attraction or repulsive ripple signals. The probability of sending attraction response is considered equal to p, and therefore $(1\ -\ p)$ probability is left for the repulsive response. Since the response of females is not determined, for simplicity, we assumed p as 50%. If the female sends an attraction signal, they will move toward each other and will mate. Considering a circle wave as illustrated in Fig. 4.a, after mating, the new position will be updated to a location between them as Fig. 4.b. If the female rejects the request, the male will mount her, then female dismount him and get him away, as shown in Fig. 4.c. The keystone may mate or be repelled, either way, the new position of keystone will be calculated by Eq. (10)

$$\begin{cases} WS_i^{t+1} = WS_i^t + R.rand & if\ mating\ happens(with\ probability\ of\ p) \\ WS_i^{t+1} = WS_i^t + R.(1 + rand) & otherwise \end{cases} \tag{10}$$

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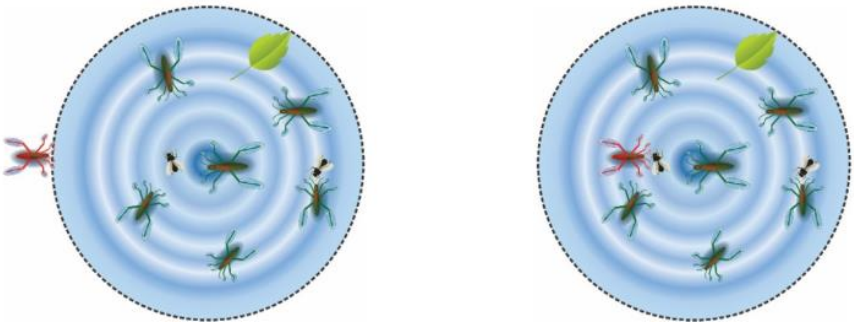
where WS_i^t is the position of i^{th} WS in the t^{th} cycle; rand is a random number between 0 and 1; R is a vector whose initial point is at the position of male (WS_i^{t-1}) and the endpoint is at the position of a female in the same territory (WS_F^{t-1}). This female can be selected by fitness proportionate selection mechanisms such as roulette wheel selection [45]. The length of R is equal to the Euclidean distance between male (WS_i^{t-1}) and female WSs (WS_F^{t-1}) (the radius of ripple wave) as Eq. (11) and Fig. 4.a.

$$R = WS_F^{t-1} - WS_i^{t-1} \tag{11}$$



(a) foraging for food if the new position is not enough rich

(b) moved toward the best strider

Figure 5: The foraging process after that the water strider could not find enough food in the new position after mating behavior

6.4. Feeding

Whether the mating happens successfully or not, the process consumes a lot of energy. Therefore, in the new position, WSs forage for food resources. As aforesaid, to assess the position for food availability, it is evaluated by the objective function. If the value of the objective function is higher than the previous state, it means that it has found the food for recovery. But if the objective value is less than the former state, it should move toward the best habitat containing the highest fitness. To this end, Eq. (12) is defined for transferring to the new position around the best WS of lake (WS_{BL}^t) containing a good deal of food resources, as shown in Fig. 5.

$$WS_i^{t+1} = WS_i^t + 2rand.\left(WS_{BL}^t - WS_i^t\right) \tag{12}$$

6.5. Death and succession

In order to determine the result of the food attainment process, the objective function is evaluated and is compared with that of the prior position. If the new fitness is lesser, the WS will die, because it not only could not find food but also increased the battle danger with the WSs of the destination territory. In this case, the newly matured larva will succeed the dead WS

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as the keystone, and the position of him is randomly initialized inside the territory as Eq. (13). If it were otherwise, the keystone would remain alive.

$$WS_i^{t+1} = Lb_j^t + 2rand\left(Ub_j^t - Lb_j^t\right) \tag{13}$$

Ub_j^t and Lb_j^t denote the maximum and minimum values of WS's position inside j^{th}territory. In other words, they determine the boundaries of died WS's territory.

6.7. Termination of WSA

In the last step of the algorithm, if the termination condition is met, the algorithm stops and reports the best-experienced position. But if the condition is not satisfied, it will return to the mating step for a new loop of the life cycle and territory establishment. In this paper, the maximum number of function evaluations ($MaxNFE$) is considered as the termination condition in all problems. However, other conditions such as the maximum number of life cycles ($MaxCycle$) can be implemented as a stopping condition of WSA.

7. DISCUSSION

In this article, multilayer perceptron neural network was used to predict the naming of nodes for the optimal wave front. This method is actually an extension of Sloan's method with neural network. Two hidden layers with 15 and 13 neurons were used in this network. This neural network was combined with Water Strider meta-heuristic algorithm and weights and biases were calculated with this algorithm. The R values as well as the comparison of output and target are given in Figure 6 for training, testing and validation data.

Quality evaluation criteria such as root mean square error (RMSE) and correlation coefficient (R) were used for evaluation. The following equations show the formulation of these criteria:

$$RMSE = \sqrt{\sum_{i=1}^{N}(z_0 - z_p)^2/N} \tag{14}$$

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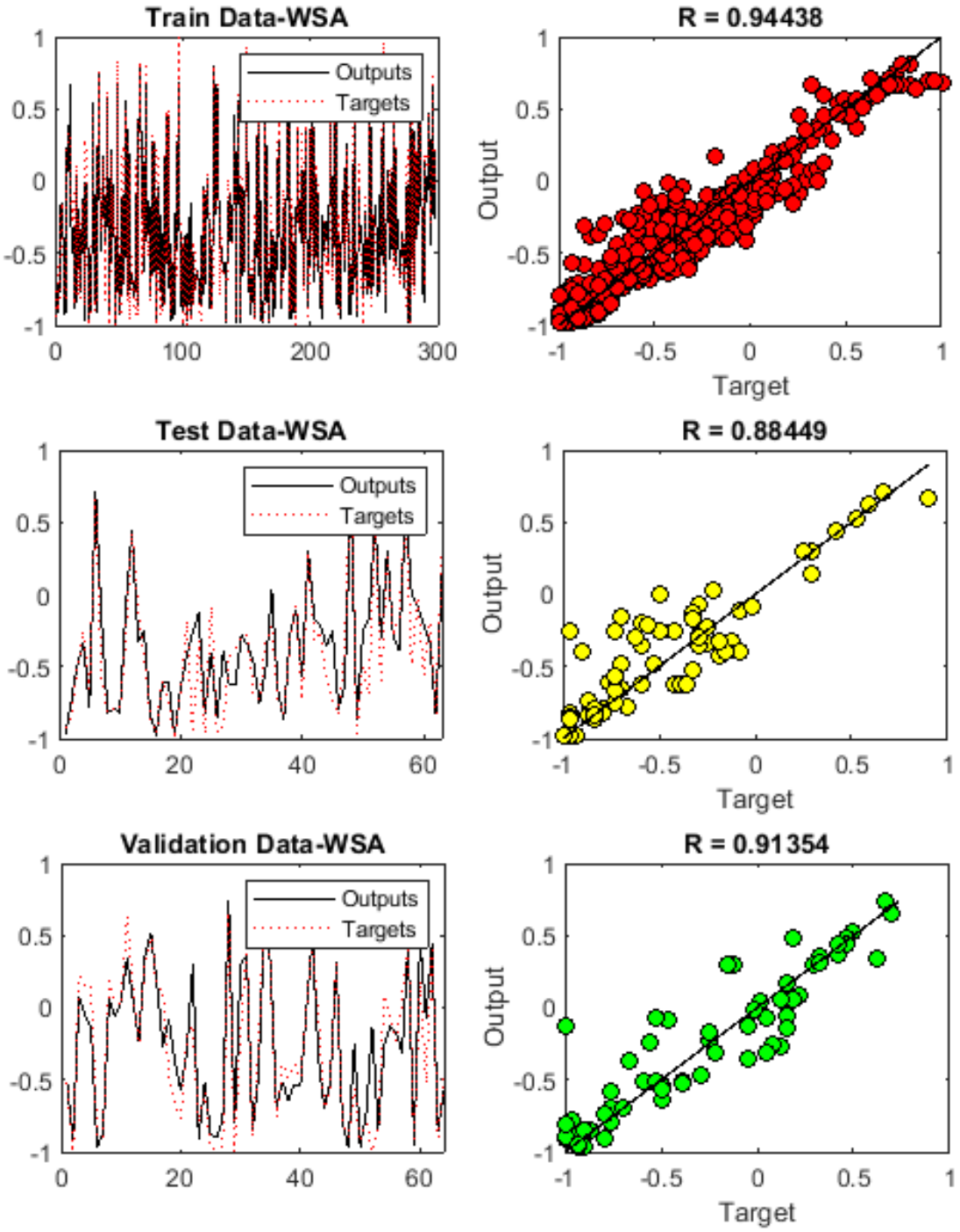

Figure 6: The result of ANN-WSA

$$R = \sqrt{1 - \frac{\sum_{i=1}^{N}(z_0 - z_p)^2}{\sum_{i=1}^{N}(z_0 - \bar{z}_i{}^2}} \tag{15}$$

For a better understanding, two examples are given in the next section and their comparison with the Sloan method.

8. EXAMPLES

In this section, three graphs are given as examples so that the root-mean-squared-error wavefront ,F_{rmse} for these examples are obtained and compared with Sloan's algorithm.

Table 2: Comparison of the results

Example	**1**	**2**	**3**
No. of nodes	224	432	464
No. of elements	800	1352	1884
F_{rmse} by Sloan	11.73	19.91	28.07
F_{rmse} by ANN-WSA	5.98	11.23	19.89

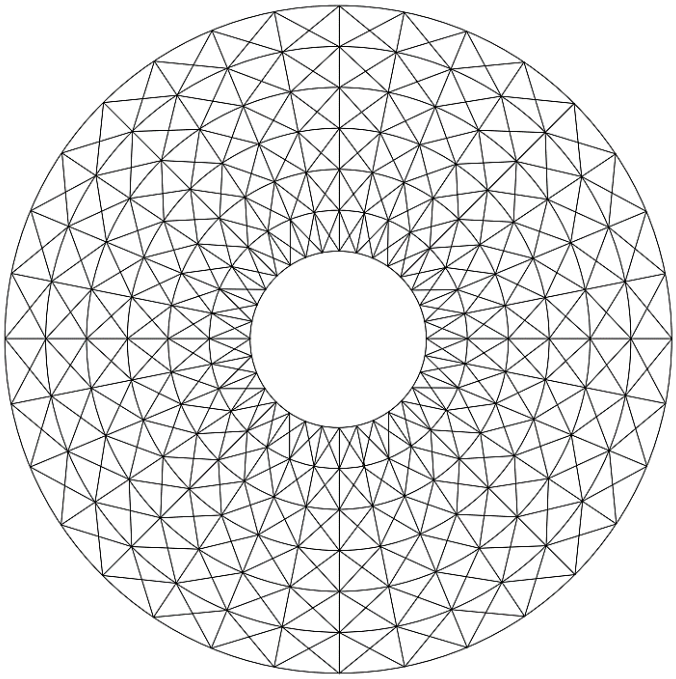

Figure 7: The graph model 1

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Example 1

A circular FEM with 224 nodes and 800 elements is considered as shown in Figure 7. For this model the application of the Sloan's method and the present hybrid method result in wavefronts provided in Table 2.

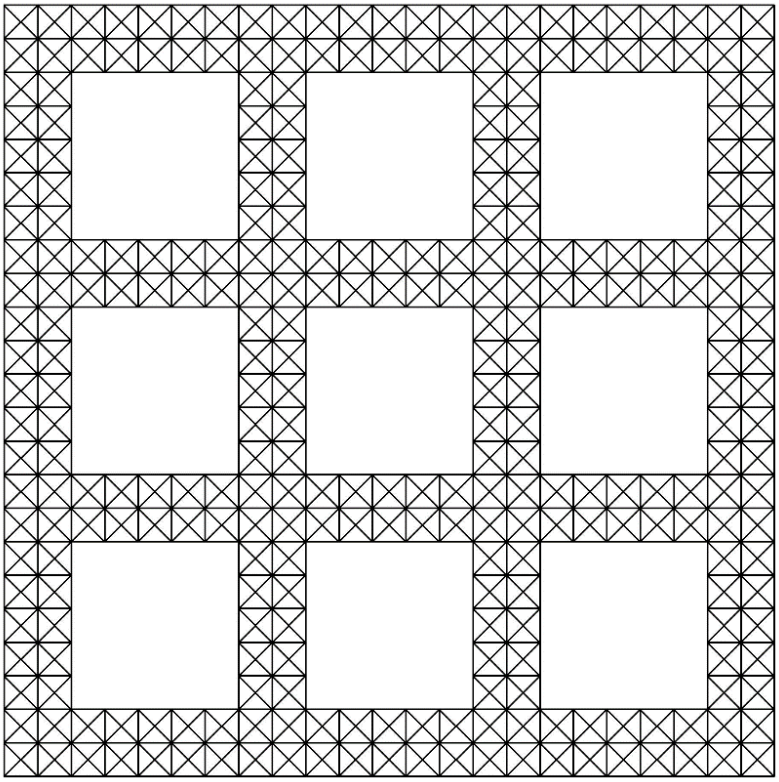

Figure 8: The graph model 2

Example 2

A FEM with 432 nodes and 1352 elements is considered as shown in Figure 8. For this model the application of the Sloan's method and the present hybrid method result in wavefronts provided in Table 2.

Example 3

A FEM with 464 nodes and 1884 elements is considered as shown in Figure 9. For this model the application of the Sloan's method and the present hybrid method result in wavefronts provided in Table 2.

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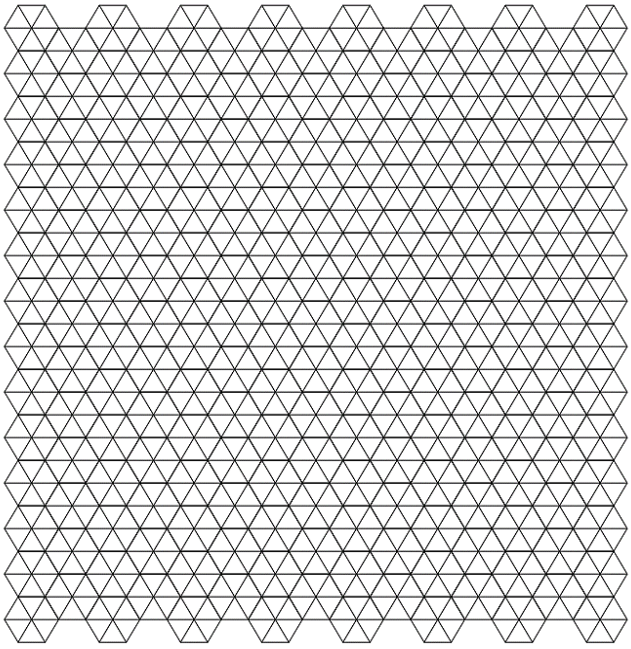

Figure 9: The graph model 3

9. CONCLUSION

A combined graph-neural-network-water strider method is presented for the nodal ordering of the graph models of FEMs, which leads to their nodal ordering with small forms for the corresponding equations. This method systematically selects the coefficients of the graph parameters for the priority function. The main goal of this paper was to combine the potentials of graph theory, artificial neural networks and Water Strider algorithm in a sorting algorithm. This method was actually a generalization and development of the Sloan model with neural network and meta-heuristic algorithm. The present method leads to reasonable wavefronts and its applications can be easily extended to the priority function with more graph parameters. The comparison of the results with three examples was investigated and it showed that the development of Sloan with neural network improves and reduces the error.

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